NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 4242

GENERAL SOLUTIONS FOR FLOW PAST SLENDER CAMBERED WINGS WITH SWEPT TRAILING EDGES AND CALCULATION OF ADDITIONAL LOADING DUE TO CONTROL SURFACES

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SUMMARY

The slender-wing-type analysis is used to obtain general expressions for the surface pressure, lift, and rolling moment for cambered wings with swept trailing edges. These results are specialized to give the additional loading due to the deflection of trailing-edge control surfaces and the loading due to a particular type of wing twist. In contrast to flat-plate wings, cambered surfaces (or wings with control surfaces) support load downstream of the maximum span of the wing where the slope of the leading edge is equal to or less than zero. The pressure distribution and the streamwise lift and rolling-moment distributions are discontinuous across the plane normal to the stream direction passing through the wing maximum span unless the slope of the wing leading edge is continuous and zero at this chordwise station. A numerical example for an aileron control is included.

INTRODUCTION

The flow field about configurations which are slender in the streamwise direction has been shown to assume a two-dimensional character in planes normal to the stream velocity. A basically three-dimensional flow is thereby approximated by solutions for a two-dimensional crossflow, and as a consequence, relatively complex configurations can be analyzed within this slender-airplane approximation. Moreover, for configurations without thickness, the solutions give the first-order approximation for the flow at transonic speeds, and the restrictions on the slenderness are not so severe as at subsonic or supersonic speeds. The slenderness concept has been applied to the analysis of a variety of configurations since the investigation of Jones (ref. 1) on the lift of low-aspect-ratio wings. The results of many of these studies are presented in references 2 and 3 together with a discussion of the underlying basis of the theory. Most of these investigations have been restricted to wings with unswept trailing edges and therefore have not had to account for the shed vortex sheet in the evaluation of the forces and surface pressures. Some exceptions are the studies of Lomax and Heaslet (ref. 4), of Mirels (ref. 5), and of Mangler (ref. 6). The indirect problem of determining the trailing-edge shape of flat-plate wings for a prescribed span loading is treated in reference 4, and the lift, pitch, and roll solutions applicable to flat-plate wings with swept trailing edges are treated in references 5 and 6.

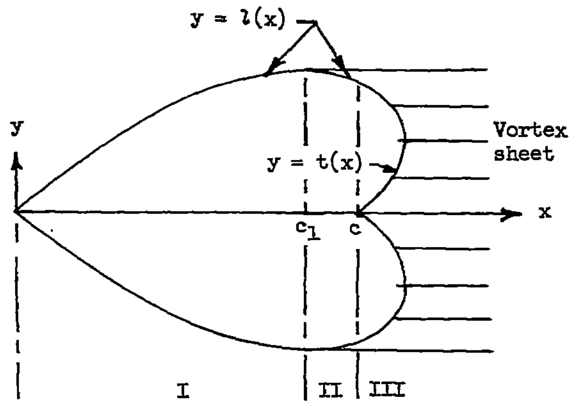
The present study is concerned with the development of general solutions for the flow past arbitrarily cambered slender wings with swept trailing edges and the subsequent application of these results to the calculation of the additional load due to the deflection of a trailing-edge control surface. The engineering usefulness of the solutions for arbitrary camber distributions is somewhat limited since the effort required to obtain numerical results is not justified in general for the slender-wing approximation; their primary value is to show some of the general features of the load distribution. The control-surface solutions as well as solutions for the loading due to pure twist, however, can be evaluated without serious difficulty.

The method of constructing the solutions is presented in section 1 and the general solutions for symmetric and antisymmetric spanwise loading are developed in sections 2 and 3. Section 4 presents the additional load due to the deflection of a flap or aileron as determined from these general equations. One example is presented to illustrate the method of solution for the effect of pure twist on the loading, and calculations are presented for the additional loading due to an aileron. A list of the more important symbols is given as appendix A.

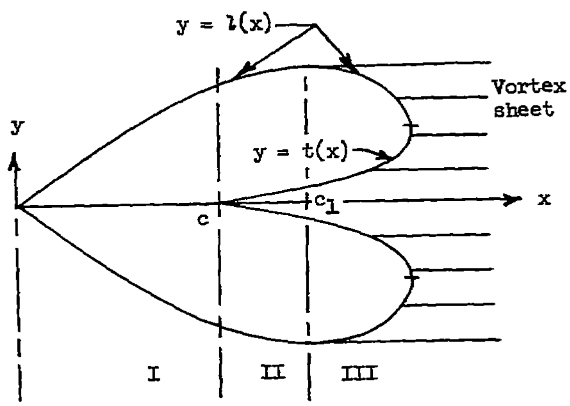
1.- EQUATIONS OF SLENDER-WING THEORY FOR LIFTING WINGS

In the slender-wing concept the spanwise extension of the wing is considered small compared with the length, and in the neighborhood of the wing the flow field assumes a two-dimensional character which for lifting wings is determined by solutions of the two-dimensional Laplace equation in planes transverse to the stream direction. The solutions for lifting wings are applicable throughout the Mach number range since the slenderness approximation causes the Mach number to vanish. Although the character of the solutions is two-dimensional, the three-dimensional aspects of the wing problem enter through the expansion or contraction of the wing in the transverse plane and through the irrotationality conditions. The Kutta condition is required to render solutions of wing problems unique whenever the trailing edge is of the subsonic type, and in the slender-wing approximation this condition is imposed on all swept trailing edges. The character of the flow in a plane normal to the stream direction depends upon the admissible behavior of the velocity at the edges of the wing. Linearized theory requires that the velocity have a square-root singularity along subsonic leading edges while the velocity is regular along the trailing edge (by the Kutta condition) and along a side edge since the load must vanish there. These requirements of linearized wing theory also apply in the slender-wing approximation.

Three regions are indicated in sketches A and B which characterize the four types of solutions for wing plan forms with swept edges. The edges $y = \pm l(x)$ and $y = \pm t(x)$ extend from the apex and the intersection of the trailing edge and root chord, respectively, to the maximum streamwise extremity of the wing. The wing semispan is denoted by s, the value of x at the intersection of the trailing edge and root chord is denoted by c, and c_1 is the value of x at the position of maximum span. The plan forms shown in sketches A and B differ in that $c > c_1$ in the former and $c < c_1$ in the latter, and the solutions in region II differ for the two plan forms. In region I the edge $l(x)$ is a leading edge and the velocity is singular there. In region II vortices are shed from the edge $l(x)$ for the wing of sketch A, and for the wing of sketch B, vortices are shed from the edge $t(x)$ while the velocity is singular on the leading edge $l(x)$. Vortices are shed from all edges in region III, and by the Kutta condition, the velocity is regular on these edges.



Sketch A



Sketch B

The slender-wing approximation affords a considerable simplification in the analysis since the governing differential equation for the disturbance velocity potential is Laplace's equation expressed in coordinates normal to the stream direction. Consequently, with the stream velocity in the direction of the x-axis, the solution can be constructed from the complex velocity $V(x;\zeta) = v(x;y,z) - iw(x;y,z)$ where $\zeta = y + iz$ is the complex variable and x occurs only as a parameter in the velocity. The quantities u, v, and w are the disturbance velocities in the x-, y-, and z-directions, respectively.

In each of the three regions (sketches A and B) the solution can be reduced to a mixed-boundary-value problem considered by Cheng and Rott (ref. 7); namely, the evaluation of a complex function in a half-space when the real and imaginary parts are prescribed alternately along segments of a line. With the mean camber surface defined by $z = Z(x,y)$ and with the subscript o used to denote values on $z = +0$, the real and imaginary parts of $V(x;\zeta)$ on $z = +0$ are alternately given in region I since $w_o = \frac{\partial Z}{\partial x}$ on the wing and v_o is zero off the wing. In regions II and III, however, the real and imaginary parts of $V(x;\zeta)$ are not alternately prescribed on $z = +0$ since v_o is not specified on the trailing vortex sheet. Differentiation of $V(x;\zeta)$ with respect to x gives a function with the desired properties since

$$V_x(x;\zeta) = v_x - iw_x$$

and by the irrotationality conditions,

$$V_x(x;\zeta) = u_y - iw_x$$

Thus the real and imaginary parts of $V_x(x;\zeta)$ are prescribed alternately on $z = +0$ since u_{o_y} vanishes everywhere off the wing, because the pressure coefficient must vanish there, and w_{o_x} is known on the wing. The prescribed values of the velocity components and their derivatives on $z = +0$ together with the requirement of vanishing velocity at infinity and the specification of the nature of the velocity at the edges are sufficient for the determination of the solution in each transverse plane.

Solution of Mixed-Boundary-Value Problem

The method of solution of the mixed-boundary-value problem for the upper half-space presented in reference 7 is based on the well-known integral relation

$$V(\zeta) = v - iw = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{w_o(\eta)\ d\eta}{\zeta - \eta} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{\zeta - \eta}\, I\left[V(\zeta)\right]_{z=+0}\, d\eta \qquad (1.1)$$

for determining the complex function $V(\zeta)$ from a knowledge of the imaginary part on $z = +0$. Consider v_o and w_o to be prescribed alternately along segments of $z = +0$. If a complex function $r(\zeta)$ can be determined, which is real on the segments where w_o is prescribed and imaginary on the segments where v_o is prescribed, then the imaginary part of the function $V(\zeta)/r(\zeta)$ is known everywhere on $z = +0$ and from equation (1.1),

$$\frac{V(\zeta)}{r(\zeta)} = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{\zeta - \eta}\, I\left[\frac{V(\zeta)}{r(\zeta)}\right]_{z=+0} d\eta + P(\zeta) \tag{1.2}$$

where I denotes the imaginary part.

The integral term in equation (1.2) does not provide a unique solution since a function $P(\zeta)$ which is real on $z = +0$ can be added without altering the prescribed conditions. Cheng and Rott (ref. 7) have shown that, with w_o prescribed on the N segments ranging from b_k to $\bar{b}_k$, the function $r(\zeta)$ is of the form

$$r(\zeta) = i\prod_{k=1}^{N}\left(\zeta - b_k\right)^{\frac{1}{2}+m_k}\left(\zeta - \bar{b}_k\right)^{\frac{1}{2}+\bar{m}_k}$$

where m_k and $\bar{m}_k$ are positive or negative integers. The order of the admissible singularities at the edges dictates the values of m_k and $\bar{m}_k$ for a particular problem. The function $P(\zeta)$ is a polynomial in ζ, the degree of which is established by the requirement that the velocity vanish at infinity. The solution for the lower half-space is known by symmetry.

Solution in Region I

The application of equation (1.2) to region I gives the usual form of the slender-wing solution. From the boundary conditions on $z = +0$ in region I, the real part of $V(x;\zeta)$ is zero for $|y| > l$ and the imaginary part is prescribed for $|y| < l$. Square-root singularities are permitted in the velocity at the edges, and the function $r(\zeta)$ is

$\frac{i}{\sqrt{\zeta^2 - l^2}}$.* The requirement of vanishing velocity at infinity dictates that $P(x;\zeta)$ is at most a function of x, and since there is no net circulation in the crossflow plane, this quantity is zero. From equation (1.2),

$$V(x;\zeta) = \frac{i}{\pi\sqrt{\zeta^2 - l^2}} \int_{-l}^{l} w_o(x;\eta)\frac{\sqrt{l^2 - \eta^2}}{\zeta - \eta}\, d\eta \tag{1.3}$$

The velocity component $v_o(x;y)$ is given by equation (1.3) for $-l \leqq y \leqq l$ on $z = +0$ and the velocity potential on the wing is obtained by integration as

$$\varphi_o(x;y) = \int_{-l}^{y} v_o(x;\eta)\, d\eta$$

$$= \frac{\sqrt{l^2 - y^2}}{\pi} \int_{-l}^{l} \frac{\nu(x;\eta)\, d\eta}{\sqrt{l^2 - \eta^2}(y - \eta)} \tag{1.4}$$

where ν is given by the indefinite integral $\nu(x;y) = \int w_o(x;\eta)\, d\eta$ since a constant can be added to ν without altering the potential.

The function $w_o(x;y)$ for $y > l$ is required to determine the solution in region II for $c < c_1$, and from equation (1.3),

$$w_o(x;y) = -\frac{1}{\pi\sqrt{y^2 - l^2}} \int_{-l}^{l} w_o(x;\eta)\frac{\sqrt{l^2 - \eta^2}}{y - \eta}\, d\eta \qquad (y > l) \tag{1.5}$$

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*Where no confusion can arise, as in equation (1.3), the argument of l (and also of the functions t and f introduced subsequently) is not explicitly indicated in the remainder of the report. The argument is either x or the streamwise variable of integration ξ. The particular quantity in question is clear from the context and is stated explicitly for the final equations.

Solution in Regions II and III

The solution in region II for $c > c_1$ is structurally similar to that in region I and is easily evaluated. The determination of the pressure in region II for $c < c_1$ as well as the solution in region III is most readily accomplished by considering separately the cases of symmetric and antisymmetric spanwise loading. For these cases, only the function $V_x(x;\zeta)$ is evaluated in this section and the pressure is determined in sections 2 and 3.

<u>Solution in region II for $c > c_1$</u>.- The trailing vortex sheet lies in the region $|y| > l$ in region II for $c > c_1$ (sketch A). The quantity w_{o_x} is known from the boundary data for $-l \leqq y \leqq l$ and u_{o_y} is zero for $|y| > l$. The Kutta condition must be satisfied on $y = \pm l$; consequently, the velocity is regular and V_x has a square-root singularity on $\pm l$. The appropriate function $r(\zeta)$ is $\dfrac{1}{\sqrt{\zeta^2 - l^2}}$, and from equation (1.2),

$$V_x(x;\zeta) = \frac{1}{\pi\sqrt{\zeta^2 - l^2}} \int_{-l}^{l} w_{o_x}(x;\eta)\sqrt{l^2 - \eta^2}\,\frac{d\eta}{\zeta - \eta} \tag{1.6}$$

The function $P(x;\zeta)$ is at most a function of x, and it can be shown readily that this function must be zero to satisfy the Kutta condition. The quantity u_{o_y} is given by the real part of equation (1.6) on $z = 0$, and since u_o is zero off the wing, the disturbance velocity on the wing is determined by integration from $-l$ to y as

$$u_o(x;y) = \frac{1}{\pi} \int_{-l}^{y} \frac{d\mu}{\sqrt{l^2 - \mu^2}} \int_{-l}^{l} w_{o_x}(x;\eta)\sqrt{l^2 - \eta^2}\,\frac{d\eta}{\mu - \eta}$$

Interchanging the order of integration and integrating by parts gives the streamwise disturbance velocity as

$$u_o(x;y) = \frac{\sqrt{l^2 - y^2}}{\pi} \int_{-l}^{l} \frac{\nu_x(x;\eta)\, d\eta}{\sqrt{l^2 - \eta^2}(y - \eta)} \tag{1.7}$$

where $\nu_x = \int w_{o_x}\, d\eta$. Comparison of equations (1.4) and (1.7) shows that $u_o(x;y)$ in region II for $c > c_1$ is given by the same integral expression as $\varphi_o(x;y)$ in region I with $\nu(x;y)$ replaced by $\nu_x(x;y)$. In region II for $c > c_1$ the slope of the edge $y = l(x)$ is less than zero and this region of the wing supports load provided w_{o_x} does not vanish everywhere in region II. For the flat-plate wing, w_{o_x} vanishes and sections downstream of the maximum span support no load. This is the result obtained by Jones (ref. 1).

<u>The function $V_x(x;\zeta)$ in region II for $c < c_1$</u>.- In region II for $c < c_1$, the trailing vortex sheet lies between the two wing panels (sketch B). The quantity u_{o_y} is zero for $|y| > l$ and $|y| < t$, and w_{o_x} is known on the wing. Square-root singularities are admissible in the velocity on the leading edge; consequently, $V_x(x;\zeta)$ has a three-halves-power singularity there while the velocity is regular on the trailing edge and $V_x(x;\zeta)$ has a square-root singularity. The appropriate function $r(\zeta)$ is $\dfrac{i}{\left(\zeta^2 - l^2\right)^{3/2}\sqrt{\zeta^2 - t^2}}$, and from equation (1.2), $V_x(x;\zeta)$ in region II for $c < c_1$ is

$$V_x(x;\zeta) = \frac{i}{\pi\left(\zeta^2-l^2\right)^{3/2}\sqrt{\zeta^2-t^2}}\left[\int_{-l}^{-t} w_{o_x}(x;\eta)\left(l^2-\eta^2\right)^{\frac{3}{2}}\sqrt{\eta^2-t^2}\,\frac{d\eta}{\zeta-\eta} - \int_{t}^{l} w_{o_x}(x;\eta)\left(l^2-\eta^2\right)^{\frac{3}{2}}\sqrt{\eta^2-t^2}\,\frac{d\eta}{\zeta-\eta} + P(x;\zeta)\right] \tag{1.8}$$

where the coefficients in the polynomial are functions of x.

<u>The function $V_x(x;\zeta)$ in region III</u>.- The solution in region III differs from that in region II for $c < c_1$ only in that the velocity is

regular on all edges in the former since the Kutta condition must be satisfied on both l and t. The function $r(\zeta)$ then is $\frac{i}{\sqrt{(\zeta^2 - l^2)(\zeta^2 - t^2)}}$, and from equation (1.2),

$$v_x(x;\zeta) = \frac{i}{\pi\sqrt{(\zeta^2-l^2)(\zeta^2-t^2)}}\left[-\int_{-l}^{-t} w_{o_x}(x;\eta)\sqrt{(l^2-\eta^2)(\eta^2-t^2)}\frac{d\eta}{\zeta-\eta} + \int_{t}^{l} w_{o_x}(x;\eta)\sqrt{(l^2-\eta^2)(\eta^2-t^2)}\frac{d\eta}{\zeta-\eta} + P(x;\zeta)\right] \quad (1.9)$$

2.- WINGS WITH SYMMETRIC SPANWISE LOADING

The expressions for the potential in region I and the disturbance velocity u_o in region II for $c > c_1$ can be specialized for either symmetric or antisymmetric spanwise loading and the linearized surface pressures are obtained directly from these quantities. In region II for $c < c_1$ and in region III it proves convenient to make use of the spanwise symmetry conditions at the outset to evaluate the surface pressures. The surface pressures for symmetric spanwise loading are developed in this section and those for antisymmetric loading are presented in section 3.

For symmetric spanwise loading $w_o(x;y) = w_o(x;-y)$, and the surface pressure coefficient in region I is determined from equation (1.4) as

$$C_p = -2\varphi_{o_x} = \frac{2}{\pi}\int_0^l w_{o_x}(x;\eta)\log\frac{1+\sqrt{\frac{l^2-y^2}{l^2-\eta^2}}}{1-\sqrt{\frac{l^2-y^2}{l^2-\eta^2}}}d\eta + \frac{4ll'}{\pi\sqrt{l^2-y^2}}\int_0^l w_o(x;\eta)\frac{d\eta}{\sqrt{l^2-\eta^2}} \quad (2.1)$$

where the prime on l denotes the derivative with respect to x. In region II for $c > c_1$ from equation (1.7),

$$C_p = -2u_o = \frac{2}{\pi}\int_0^l w_{o_x}(x;\eta)\log\frac{1 + \sqrt{\frac{l^2 - y^2}{l^2 - \eta^2}}}{1 - \sqrt{\frac{l^2 - y^2}{l^2 - \eta^2}}}d\eta \quad (2.2)$$

From these equations the pressure is seen to be discontinuous at the boundary of regions I and II for $c > c_1$ if $l'(c_1)$ is different from zero.

Solution in Region II for $c < c_1$

Pressure.- For symmetric spanwise loading the integration of equation (1.8) can be carried out over the right wing panel to give

$$V_x(x;\zeta) = \frac{i}{\pi\left(\zeta^2-l^2\right)^{3/2}\sqrt{\zeta^2-t^2}}\left[P(x;\zeta)-2\int_t^l w_{o_x}(x;\eta)\left(l^2-\eta^2\right)^{\frac{3}{2}}\sqrt{\eta^2-t^2}\,\frac{\eta\, d\eta}{\zeta^2-\eta^2}\right] \tag{2.3}$$

The symmetry condition requires that the polynomial be an even function of ζ, and for the velocity to vanish at infinity, $P(x;\zeta) = A + B\zeta^2$ where A and B are functions of x. The quantity u_{o_y} is given by equation (2.3) for $t \leqq y \leqq l$, and since u_o is zero off the wing, the disturbance velocity on the wing for $t \leqq y \leqq l$ is

$$u_o(x;y) = \int_l^y u_{o_\eta}\, d\eta$$

and after some reduction

$$u_o(x;y) = -\frac{1}{\pi l\left(l^2-t^2\right)}\left[(A+Bl^2)E(\phi,k)-(A+Bt^2)F(\phi,k)+(A+Bl^2)\frac{y}{l}\sqrt{\frac{y^2-t^2}{l^2-y^2}}-G(x;y)\right] \tag{2.4}$$

where $F(\phi,k)$ and $E(\phi,k)$ are incomplete elliptic integrals of the first and second kind, respectively, $\sin\phi = \sqrt{\dfrac{l^2 - y^2}{l^2 - t^2}}$, the modulus k is given by $k^2 = 1 - \frac{t^2}{l^2}$, and

$$G(x;y) = \frac{2}{\pi} \text{⨍}_y^l \frac{d\mu}{\left(l^2 - \mu^2\right)^{3/2}\sqrt{\mu^2 - t^2}} \int_t^l w_{o_x}(x;\eta)\left(l^2 - \eta^2\right)^{\frac{3}{2}}\sqrt{\eta^2 - t^2}\,\frac{\eta\, d\eta}{\mu^2 - \eta^2} \tag{2.5}$$

The sign ⨍ denotes the finite part of an integral. Satisfying the Kutta condition at $y = t$ gives the streamwise disturbance velocity as

$$u_o(x;y) = -\frac{A + Bl^2}{\pi l\left(l^2 - t^2\right)}\left[Z(\phi,k) + \frac{y}{l}\sqrt{\frac{y^2 - t^2}{l^2 - y^2}}\right] + \frac{F(\phi,k)}{K}\,G(x;t) - G(x;y) \tag{2.6}$$

where $Z(\phi,k) = E(\phi,k) - \frac{E}{K}F(\phi,k)$ is the Jacobian zeta function.

The functions $G(x;y)$ and $G(x;t)$ are evaluated in appendix B. These functions can be reduced to a single integral involving elliptic integrals and related functions. The combination of these two functions which occurs in equation (2.6) can be expressed in terms of the quantity in brackets in equation (2.6) and a theta function. With the use of equation (B4) the pressure coefficient in region II for $c < c_1$ then becomes

$$C_p = -2S(x)l'\left[Z(\phi,k) + \frac{y}{l}\sqrt{\frac{y^2 - t^2}{l^2 - y^2}}\right] + \frac{4}{\pi}\int_t^l w_{o_x}(x;\eta)\Omega(x;y,\eta)\, d\eta \tag{2.7}$$

where the unknown quantity $S(x)$ which is related to A and B by

$$S(x)l' = \frac{1}{\pi l\left(l^2 - t^2\right)}\left[2\int_t^l w_{o_x}(x;\eta)\sqrt{\left(l^2 - \eta^2\right)\left(\eta^2 - t^2\right)}\eta\, d\eta - \left(A + Bl^2\right)\right]$$

is introduced to facilitate the analysis. The function $\Omega(x;y,\eta)$ is expressed in terms of an infinite series by equation (B3). Equation (2.7) is equivalent to that obtained for the pressure coefficient for the flat-plate wing in reference 5 with the addition of the integral involving $\Omega(x;y,\eta)$ and the integral in the definition of $S(x)$.

Integral equation for $S(x)$.- The Kutta condition gives one relation between A and B and an additional relation is required to evaluate these functions, or equivalently, the function $S(x)$. Since the solution has been constructed from w_{o_x} whereas the surface boundary condition is in terms of w_o, the second relation is that the solution must give the prescribed w_o on the wing. The function $S(x)$ is determined by a procedure essentially the same as that used by Mirels (ref. 5). First, w_o on the wing is expressed as

$$w_o(x;y) = \int_{-\infty}^{x} w_{o_\xi}(\xi;y)\, d\xi$$

where it is essential that y be greater than $l(c)$ for the path of integration, since for $y < l(c)$, the relation is an identity and gives no condition on $S(x)$. Separating the integration into the three parts $-\infty$ to c, c to $l^{-1}(y)$, and $l^{-1}(y)$ to x gives the equation

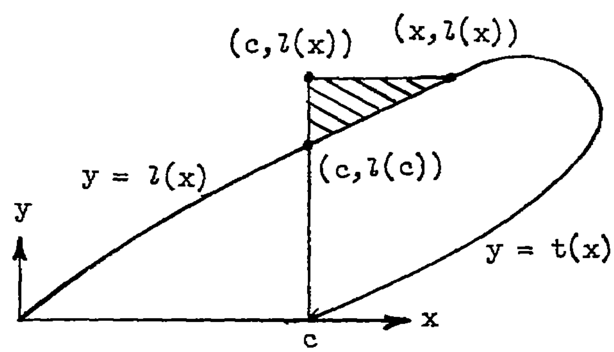
$$w_o\left(l^{-1}(y);y\right) - w_o(c;y) = \int_c^{l^{-1}(y)} w_{o_\xi}(\xi;y)\, d\xi \tag{2.8}$$

where $w_o\left(l^{-1}(y);y\right)$ is the prescribed value of w_o along the leading edge and $w_o(c;y)$ is the value of the downwash off the wing in region I and is given by equation (1.5) with $x = c$.

Equation (2.8) provides an integral equation for the determination of $S(x)$ but involves both the unknown function $S(x)$ and dS/dx; the function dS/dx arises in the course of evaluating the finite part of the integral. An equation in terms of $S(x)$ only is obtained by integrating equation (2.8) from $y = l(c)$ to $y = l(x)$. Then,

$$\int_{l(c)}^{l(x)} \left[w_o\left(l^{-1}(y);y\right) - w_o(c;y)\right] dy = \int_c^x d\xi \int_l^{l(x)} w_{o_\xi}(\xi;y)\, dy \tag{2.9}$$

where the order of integration of the double integral has been interchanged. The region of integration of the right-hand member of equation (2.9) is shown shaded in sketch C.



Sketch C

The expression for w_{o_ξ} appearing in the integral of equation (2.9) is obtained from equation (2.3) for $y > l$. Satisfying the Kutta condition from equation (2.4) gives a relation between the functions A and B, and the polynomial can be expressed in the form

$$P(x;\zeta) \equiv A + B\zeta^2 = -\pi l\left(\zeta^2 - l^2\right)\left[\frac{A + Bl^2}{\pi l\left(l^2 - t^2\right)}\left(\frac{E}{K} - \frac{\zeta^2 - t^2}{\zeta^2 - l^2}\right) + \frac{G(x;t)}{K}\right]$$

where K and E are the complete elliptic integrals of the first and second kind, respectively, with modulus k. Then, when the integration is performed, the inner integral in equation (2.9) becomes

$$\int_l^{l(x)} w_{o_\xi}\, dy = g(\xi;l(x)) + \frac{F(\theta,k')}{K} G(\xi;t) + \frac{A + Bl^2}{\pi l\left(l^2 - t^2\right)}\left[\frac{\pi}{2}\frac{\Lambda_o(\theta,k)}{K} + \frac{k^2 l\; l(x)}{\sqrt{l^2(x) - l^2}\sqrt{l^2(x) - t^2}}\right] \tag{2.10}$$

where $\Lambda_o(\theta,k)$ is Heuman's lambda function (ref. 10), k' is the complementary modulus, and

$$k' = \sqrt{1 - k^2} = \frac{t}{l} \qquad \sin\theta = \sqrt{\frac{l^2(x) - l^2}{l^2(x) - t^2}}$$

The function $G(x;t)$ is given by equation (2.5) with $y = t$ and

$$g(\xi;l(x)) = \frac{2}{\pi}\int_l^{l(x)} \frac{dy}{\left(y^2-l^2\right)^{3/2}\sqrt{y^2-t^2}} \int_t^l w_{o_\xi}(\xi;\eta)\left(l^2-\eta^2\right)^{\frac{3}{2}}\sqrt{\eta^2-t^2}\,\frac{\eta\,d\eta}{y^2-\eta^2} \tag{2.11}$$

The functions $G(\xi;t)$ and $g(\xi;l(x))$ are evaluated in appendix B. These functions can be reduced to a single integral with the integrand expressed in terms of elliptic integrals and related functions. The combination of the two terms which arises in equation (2.10) can be expressed in terms of the quantity in brackets in equation (2.10), a theta function, and elliptic integrals. From equations (2.10) and (B6),

$$\int_l^{l(x)} w_{o_\xi}(\xi;y)\,dy = H(\xi;l(x)) - S(\xi)l'(\xi)\left[\frac{\pi}{2}\,\frac{\Lambda_o(\theta,k)}{K} + \frac{k^2 l\; l(x)}{\sqrt{l^2(x)-l^2}\sqrt{l^2(x)-t^2}}\right] \tag{2.12}$$

where

$$H(\xi;l(x)) = \int_t^l w_{o_\xi}(\xi;\eta)\left[\frac{F(\theta,k')F(\omega,k)}{K'K} - \frac{\Gamma(\xi;\eta)}{K'}\right]d\eta \tag{2.13}$$

In equation (2.13), K' is the complete elliptic integral of the first kind with the complementary modulus k' and $\sin\omega = \sqrt{\frac{l^2 - \eta^2}{l^2 - t^2}}$. The function $\Gamma(\xi;\eta)$ is evaluated from the series given by equation (B9).

From equations (2.9) and (2.12) the final form of the integral equation for $S(x)$ is

$$M(x) = \int_c^x S(\xi)l'(\xi)\left[\frac{\pi}{2}\frac{\Lambda_0(\theta,k)}{K} + \frac{k^2 l\ l(x)}{\sqrt{l^2(x) - l^2}\sqrt{l^2(x) - t^2}}\right]d\xi \qquad (2.14)$$

where the left-hand member is

$$M(x) = \int_{l(c)}^{l(x)}\left[w_o(c;y) - w_o\left(l^{-1}(y);y\right)\right]dy + \int_c^x H(\xi;l(x))\,d\xi \qquad (2.15)$$

The quantity $w_o(c;y)$ is given by equation (1.5) with $x = c$, and $w_o\left(l^{-1}(y);y\right)$ is the prescribed value of w_o along the leading edge. In equations (2.13) and (2.14) l, t, and k are functions of the variable of integration ξ where the argument is not explicitly indicated. The kernel of equation (2.14) is equivalent to that given by Mirels for the flat-plate wing at angle of attack α. For the flat plate, w_{o_x} vanishes; hence, the function $H(\xi;l(x))$ and the integral in equation (2.5) are zero and $M(x) = \alpha\sqrt{l^2(x) - l^2(c)}$.

Equations (2.7) and (2.14) together determine the pressure in region II for $c < c_1$. Equation (2.14) can be inverted by numerical methods to obtain $S(\xi)$ by a method similar to that of reference 5. (See appendix C.) The Jacobian zeta function and the Heuman lambda function are tabulated functions (refs. 8, 9, and 10, for example). The functions $\Omega(x;y,\eta)$ and $\Gamma(\xi;\eta)$ which arise in the equations for the pressure coefficient and $M(x)$, respectively, are determined from an infinite series. These series converge rapidly, however, and the evaluation of $\Omega(x;y,\eta)$ and $\Gamma(\xi;\eta)$ offers no serious difficulty.

Calculation of the pressure for x near c.- The integral equation (2.14) and the expression for the pressure coefficient (eq. (2.7)) can be approximated to give a somewhat simplified as well as useful solution for x near c. In this neighborhood, the kernel of the integral equation can be approximated and the inversion for $S(x)$ can be obtained in a closed form. The integral equation is approximated by

$$M(x) = \int_c^x \frac{l l'}{\sqrt{l^2(x) - l^2}} \left[S(\xi)k - \frac{4}{\gamma} S(c) \int_0^{\frac{k'}{4}} \frac{d\mu}{\log \mu} \right] d\xi$$

which is equivalent to that given in reference 5. The inner integral is the logarithmic integral (tabulated in ref. 11, for example) and $\gamma = \left.\frac{dt}{dl}\right|_{x=c}$. The quantity in brackets is independent of x, and the inversion of this integral equation of the Abel type is

$$S(x)l(x)l'(x) = \frac{1}{k}\left[\frac{4S(c)}{\gamma} l(x)l'(x) \int_0^{\frac{k'}{4}} \frac{d\mu}{\log \mu} + \frac{2}{\pi} \frac{d}{dx} \int_c^x \frac{M(\xi) l l' \, d\xi}{\sqrt{l^2(x) - l^2}}\right] \tag{2.16}$$

where k and k' are functions of x. From equation (B11) the function $H(\xi;l(x))$ for x near c is

$$H(\xi;l(x)) = -\frac{2}{\pi}\sqrt{\frac{l^2(x) - l^2}{l^2(x) - t^2}} \int_t^l w_{o_\xi}(\xi;\eta) \left[\frac{\eta}{l}\sqrt{\frac{\eta^2 - t^2}{l^2 - \eta^2}} + Z(\omega,k)\right] d\eta \tag{2.17}$$

The function $\Omega(x;y,\eta)$ which arises in the evaluation of the pressure is approximated by the first term of the series representation as

$$\left.\begin{aligned} \Omega(x;\eta) &= \frac{1}{2} \log \frac{\sin(\sigma + \tau)}{\sin(\sigma - \tau)} && (x \text{ near } c) \\ \Omega(x;\eta) &= \frac{1}{2} \log \frac{1 + \sqrt{\dfrac{l^2 - y^2}{l^2 - \eta^2}}}{1 - \sqrt{\dfrac{l^2 - y^2}{l^2 - \eta^2}}} && (x = c) \end{aligned}\right\} \tag{2.18}$$

where

$$\sigma = \frac{\pi}{2} \frac{F(\omega,k)}{K} \qquad\qquad \tau = \frac{\pi}{2} \frac{F(\phi,k)}{K}$$

It can be shown that $S(c)$ depends on the camber distribution at $x = c$ through the relation

$$S(c) = -\frac{2}{\pi} \int_0^{l(c)} \frac{w_o(c;\eta)\, d\eta}{\sqrt{l^2(c) - \eta^2}}$$

and since $Z(\phi,1) = \frac{1}{l(c)} \sqrt{l^2(c) - y^2}$, comparison of equations (2.1) and (2.7) shows that the pressure is continuous across the boundary of regions I and II for $c < c_1$ if $l'(c)$ is continuous.

Solution in Region III

For symmetric spanwise loading equation (1.9) can be written as

$$V_x(x;\zeta) = \frac{i}{\pi\sqrt{\left(\zeta^2-l^2\right)\left(\zeta^2-t^2\right)}}\left[P(x;\zeta) + 2\int_t^l w_{o_x}(x;\eta)\sqrt{\left(l^2-\eta^2\right)\left(\eta^2-t^2\right)}\, \frac{\eta\, d\eta}{\zeta^2-\eta^2}\right] \tag{2.19}$$

The condition of symmetry and the requirement of vanishing velocity at infinity determine $P(x;\zeta) = A$ where A is a function of x only. The quantity u_{o_y} is given by the real part of equation (2.19), and integration gives

$$u_o(x;y) = -\frac{2}{\pi}\int_y^l \frac{d\mu}{\sqrt{\left(l^2-\mu^2\right)\left(\mu^2-t^2\right)}} \int_t^l w_{o_x}(x;\eta)\sqrt{\left(l^2-\eta^2\right)\left(\eta^2-t^2\right)}\,\frac{\eta\, d\eta}{\mu^2-\eta^2} - \frac{A}{\pi}\int_y^l \frac{d\mu}{\sqrt{\left(l^2-\mu^2\right)\left(\mu^2-t^2\right)}}$$

Satisfying the Kutta condition at $y = t$ determines A. Interchanging the order of integration and integrating gives the pressure coefficient in region III

$$C_p = -\frac{4}{\pi l}\int_t^l w_{o_x}(x;\eta)\sqrt{\frac{\eta^2 - t^2}{l^2 - \eta^2}}\left[\Pi(\theta,\alpha^2,k) - \frac{F(\theta,k)}{K}\,\Pi\left(\frac{\pi}{2},\alpha^2,k\right)\right]\eta\ d\eta$$

$$= \frac{4}{\pi}\int_t^l w_{o_x}(x;\eta)\Omega(x;y,\eta)\ d\eta \qquad (2.20)$$

where $\Pi(\theta,\alpha^2,k)$ is the elliptic integral of the third kind with the parameter $\alpha^2 = \dfrac{l^2 - t^2}{l^2 - \eta^2}$; the last form of equation (2.20) follows from the substitution of equations (B2) for the elliptic integrals. From equation (2.20) the pressure coefficient in region III is zero only if w_{o_x} is zero. Comparison of equation (2.20) with equations (2.1) and (2.7) shows that the pressure across the boundary of regions II and III is discontinuous for $c < c_1$ if $l'(c_1) \neq 0$ and is continuous for $c > c_1$. Thus, for either $c > c_1$ or $c < c_1$, the surface pressure is discontinuous across the plane normal to the stream direction which passes through the wing maximum span unless l' is continuous and zero at $x = c_1$. Further, the surface pressure is continuous across the plane passing through the root chord if $l'(c)$ is continuous.

Lift

The lift can be determined from the asymptotic form of $V(x;\zeta)$. For a wing, the function $V(x;\zeta)$ for large values of ζ is of the form

$$V(x;\zeta) = -\sum_{m=1} m a_m \zeta^{-(m+1)} \qquad (2.21)$$

From reference 5 or 12 the lift carried by the wing upstream of x is

$$L = I(4\pi q_\infty a_1)$$

where q_∞ is the stream dynamic pressure.

The coefficient a_1 in region I is determined from equation (1.3); thus L becomes

$$L = -4q_\infty \int_{-l}^{l} w_o(x;\eta)\sqrt{l^2 - \eta^2}\, d\eta \tag{2.22}$$

In region II for $c > c_1$ the function da_1/dx is obtained from equation (1.6), and dL/dx is then determined as

$$\frac{dL}{dx} = -4q_\infty \int_{-l}^{l} w_{o_x}(x;\eta)\sqrt{l^2 - \eta^2}\, d\eta \tag{2.23}$$

Similarly, the function dL/dx in region II for $c < c_1$ is determined from equation (2.3) as

$$\frac{dL}{dx} = 4\pi q_\infty l\left[S(x)l'(x)\left(1 - \frac{E}{K}\right) - \frac{2}{\pi}\int_{t}^{l} w_{o_x}(x;\eta)Z(\omega,k)\, d\eta\right] \tag{2.24}$$

From equation (2.19), dL/dx in region III is

$$\frac{dL}{dx} = -8q_\infty l \int_{t}^{l} w_{o_x}(x;\eta)Z(\omega,k)\, d\eta \tag{2.25}$$

and the total lift is obtained by integration.

The lift supported by the part of the wing in region I is dependent only upon $w_o(c;y)$ (or $w_o(c_1;y)$ for $c > c_1$). In regions II and III, however, the contribution to the lift depends upon the camber distribution throughout these regions.

Wings With Twist

The slender-wing solutions assume their simplest form for the flat plate and for wings with twist but no camber. For these conditions the wing supports load only where $l' > 0$. The flat-plate solution has been given in references 5 and 6, and the solution for twist proportional to y^2 is developed here to illustrate the use of the equations.

Let $w_o(x;y) = \frac{a}{l^2(c)} y^2$ where a is a constant. Then from equation (2.1), the surface pressure coefficient for $x \leqq c$ is

$$C_p = -2\varphi_{o_x} = \frac{a}{l^2(c)} \frac{l^3 l'}{\sqrt{l^2 - y^2}}$$

and from equation (2.22) the lift carried by the part of the wing upstream of x in region I is

$$L = -\frac{\pi}{2} a q_\infty \frac{l^4}{l^2(c)}$$

The loading is zero in regions II and III except for region II with $c < c_1$. The solution in region II with $c < c_1$ requires the evaluation of the function $S(x)$. Only the terms $w_o(c;y)$ and $w_o\left(l^{-1}(y);y\right)$ contribute to the left-hand member of the integral equation for $S(x)$; from equation (1.5),

$$w_o(c;y) = \frac{a}{2l^2(c)\sqrt{y^2 - l^2(c)}}\left[2y^3 - yl^2(c) - 2y^2\sqrt{y^2 - l^2(c)}\right]$$

and $w_o\left(l^{-1}(y);y\right) = \frac{a^2}{l^2(c)} y^2$. Then from equation (2.15), the left-hand member of the integral equation is

$$M(x) = -\frac{a}{6l^2(c)}\left[2l^2 + l^2(c)\right]\sqrt{l^2 - l^2(c)}$$

The inversion of equation (2.14) for x near c is given by equation (2.16) as

$$S(x) = -\frac{a}{2k}\left[\frac{4}{\gamma}\int_0^{\frac{k'}{4}}\frac{d\mu}{\log\mu} + \frac{l^2(x)}{l^2(c)}\right] \qquad (2.26)$$

and $S(c) = -\frac{a}{2}$. From equation (2.7) the surface pressure coefficient for $c < x < c_1$ is

$$C_p = -2S(x)l'\left[Z(\phi,k) + \frac{y}{l}\sqrt{\frac{y^2 - t^2}{l^2 - y^2}}\right]$$

and at $x = c$,

$$C_p = \frac{all'}{\sqrt{l^2 - y^2}}$$

From equation (2.23) the lift per unit length for $c < x < c_1$ is

$$\frac{dL}{dx} = 4\pi q_\infty S(x)ll'\left(1 - \frac{E}{K}\right)$$

and at $x = c$,

$$\frac{dL}{dx} = 2\pi q_\infty all'$$

Comparison with the expressions for the lift and the pressure coefficient in region I shows that dL/dx and C_p are continuous across the boundary of regions I and II if l' is continuous at $x = c$.

The function $S(x)/a$ computed from equation (2.26) is compared in figure 1 with that calculated by the numerical method given in appendix C for a wing with straight edges ($l = m_l x$ and $t = m_t(x-c)$) for $\gamma = 1$, 2, and 3. The interval for the numerical method was taken as 0.05, and Simpson's rule was used to evaluate the integrals. The results show that the approximate inversion provides a very satisfactory solution even when k differs appreciably from unity.

3.- WINGS WITH ANTISYMMETRIC SPANWISE LOADING

The pressures in region I and region II for $c > c_1$ for antisymmetric spanwise loading follow directly from equations (1.4) and (1.7). The pressure in region II for $c < c_1$ and the pressure in region III can be determined in a manner similar to that for the case of symmetric loading.

For antisymmetric spanwise loading, $w_o(x;y) = -w_o(x;-y)$ and the surface pressure coefficient in region I is determined from equation (1.4) as

$$C_p = -2\varphi_{o_x} = \frac{2}{\pi}\int_0^l w_{o_x}(x;\eta)\log\left|\frac{1+\frac{\eta}{y}\sqrt{\frac{l^2-y^2}{l^2-\eta^2}}}{1-\frac{\eta}{y}\sqrt{\frac{l^2-y^2}{l^2-\eta^2}}}\right|d\eta + \frac{4}{\pi}\frac{yl'}{l\sqrt{l^2-y^2}}\int_0^l \frac{w_o(x;\eta)\eta\,d\eta}{\sqrt{l^2-\eta^2}} \tag{3.1}$$

In region II for $c > c_1$ from equation (1.7),

$$C_p = -2u_o = \frac{2}{\pi}\int_0^l w_{o_x}(x;\eta)\log\left|\frac{1+\frac{\eta}{y}\sqrt{\frac{l^2-y^2}{l^2-\eta^2}}}{1-\frac{\eta}{y}\sqrt{\frac{l^2-y^2}{l^2-\eta^2}}}\right|d\eta \tag{3.2}$$

Thus the pressure is discontinuous at $x = c_1$, the boundary of regions I and II for $c > c_1$, if $l'(c_1)$ is different from zero.

Solution in Region II for $c < c_1$

Pressure in region II.- For antisymmetric spanwise loading, equation (1.6) becomes

$$V_x(x;\zeta) = \frac{-i}{\pi\left(\zeta^2-l^2\right)^{3/2}\sqrt{\zeta^2-t^2}}\left[2\zeta\int_t^l w_{o_x}(x;\eta)\left(l^2-\eta^2\right)^{\frac{3}{2}}\sqrt{\eta^2-t^2}\,\frac{d\eta}{\zeta^2-\eta^2} + P(x;\zeta)\right] \tag{3.3}$$

The symmetry condition requires that the polynomial be an odd function of ζ, and since the velocity must vanish at infinity, $P(x;\zeta) = A\zeta + B\zeta^3$ where the coefficients A and B are functions of x only. The quantity u_{o_y} is given by the real part of equation (3.3), and since u_o is zero off the wing, the disturbance velocity on the right wing panel is

$$u_o(x;y) = \int_l^y u_{o_\mu}(x;\mu)\,d\mu$$

$$= \frac{A}{\pi\left(l^2-t^2\right)}\sqrt{\frac{y^2-t^2}{l^2-y^2}} + \frac{B}{\pi}\left(\sin^{-1}\sqrt{\frac{l^2-y^2}{l^2-t^2}} + \frac{l^2}{l^2-t^2}\sqrt{\frac{y^2-t^2}{l^2-y^2}}\right) -$$

$$\frac{2}{\pi}\int_y^l \frac{\mu\,d\mu}{\left(l^2-\mu^2\right)^{3/2}\sqrt{\mu^2-t^2}}\int_t^l w_{o_x}(x;\eta)\left(l^2-\eta^2\right)^{\frac{3}{2}}\sqrt{\eta^2-t^2}\,\frac{d\eta}{\mu^2-\eta^2}$$

With an interchange of the order of integration and after integration on μ, it can be seen that the Kutta condition requires that B equal zero. The linearized expression for the pressure coefficient on the right wing panel then becomes

$$C_p = -2u_o = \frac{2}{\pi}\int_t^l w_{o_x}(x;\eta)\log\frac{1+\sqrt{\frac{(l^2-\eta^2)(y^2-t^2)}{(l^2-y^2)(\eta^2-t^2)}}}{1-\sqrt{\frac{(l^2-\eta^2)(y^2-t^2)}{(l^2-y^2)(\eta^2-t^2)}}}d\eta - 2Q(x)ll'\sqrt{\frac{y^2-t^2}{l^2-y^2}} \tag{3.4}$$

and is of the opposite sign on the left panel. The function $Q(x)$, which is related to the coefficient A by

$$Q(x)ll' = \frac{1}{\pi(l^2 - t^2)}\left[A + 2\int_t^l w_{o_x}(x;\eta)\sqrt{(l^2 - \eta^2)(\eta^2 - t^2)}\,d\eta\right]$$

is introduced for convenience.

Integral equation for $Q(x)$.- The function $Q(x)$ can be determined from an integral relation in a manner similar to that used to evaluate $S(x)$. The procedure is to integrate w_{o_x} over the shaded region in sketch C. Then the integral relation is

$$\int_{l(c)}^{l(x)}\left[w_o\left(l^{-1}(y);y\right) - w_o(c;y)\right]dy = \int_c^x d\xi \int_l^{l(x)} w_{o_\xi}(\xi;y)\,dy \tag{3.5}$$

where $w_o\left(l^{-1}(y);y\right)$ is the prescribed value of w_o on the leading edge and $w_o(c;y)$ is given by equation (1.5) with $x = c$. Substituting w_{o_ξ} from equation (3.3) into equation (3.5) and integrating on y gives the integral equation for $Q(x)$

$$N(x) = \int_c^x Q(\xi)ll'\sqrt{\frac{l^2(x) - t^2}{l^2(x) - l^2}}\,d\xi \tag{3.6}$$

where the left-hand member is

$$N(x) = \int_{l(c)}^{l(x)} \left[w_o(c;y) - w_o\left(l^{-1}(y);y\right) \right] dy + \int_c^x h(\xi;l(x))\, d\xi \qquad (3.7)$$

The quantity $w_o(c;y)$ is given by equation (1.5) with $x = c$, $w_o\left(l^{-1}(y);y\right)$ is the prescribed value of w_o along the leading edge, and

$$h(\xi;l(x)) = -\frac{2}{\pi} \int_t^l w_{o_\xi}(\xi;\eta) \tan^{-1}\sqrt{\frac{\left[l^2(x) - l^2\right]\left(\eta^2 - t^2\right)}{\left[l^2(x) - t^2\right]\left(l^2 - \eta^2\right)}}\, d\eta \qquad (3.8)$$

In equations (3.4), (3.6), and (3.8) the argument of l and t is ξ except where indicated explicitly.

Mirels' solution of the flat-plate rolling wing is a particular example of these equations. For the rolling wing $w_o = -\omega_x y$, where ω_x is the angular velocity, and since w_o is independent of x, the function $h(\xi;l(x))$ vanishes and $N(x)$ becomes $\frac{1}{2}\omega_x l\sqrt{l^2 - l^2(c)}$.

The numerical solution of equation (3.6) is readily determined by the method of reference 3 once the function $N(x)$ is evaluated. (See appendix C.) For x near c, t is small compared with l and can be neglected, and the solution is given in closed form as

$$Q(x)l(x)l'(x) = \frac{2}{\pi}\frac{d}{dx}\int_c^x \frac{N(\xi)l'\, d\xi}{\sqrt{l^2(x) - l^2}} \qquad (3.9)$$

The function $Q(c)$ can be shown to depend upon the camber distribution at $x = c$ through the relation

$$Q(c) = -\frac{2}{\pi l^2(c)} \int_0^{l(c)} \frac{w_o(c;\eta)\eta\, d\eta}{\sqrt{l^2(c) - \eta^2}} \quad .$$

Solution in Region III

For antisymmetric spanwise loading, equation (1.9) can be written as

$$V_x(x;\zeta) = \frac{i}{\pi\sqrt{\left(\zeta^2-l^2\right)\left(\zeta^2-t^2\right)}}\left[2\zeta\int_t^l w_{o_x}(x;\eta)\sqrt{\left(l^2-\eta^2\right)\left(\eta^2-t^2\right)}\,\frac{d\eta}{\zeta^2-\eta^2} + A\zeta\right] \tag{3.10}$$

where the polynomial is at most $A\zeta$ from the symmetry conditions and the requirement of vanishing velocity at infinity. For $t \leqq y \leqq l$ on $z = +0$, equation (3.10) gives u_{o_y}, and u_o on the right wing panel is obtained by integration from l to y as

$$u_o(x;y) = -\frac{2}{\pi}\int_y^l \frac{\mu\,d\mu}{\sqrt{\left(l^2-\mu^2\right)\left(\mu^2-t^2\right)}}\int_t^l w_{o_x}(x;\eta)\sqrt{\left(l^2-\eta^2\right)\left(\eta^2-t^2\right)}\,\frac{d\eta}{\zeta^2-\eta^2}$$

where A is determined as zero by the Kutta condition at $y = t$. With the use of the linearized pressure relation and after an interchange of the order of integration the pressure coefficient for $t \leqq y \leqq l$ becomes

$$C_p = \frac{2}{\pi}\int_t^l w_{o_x}(x;\eta)\log\frac{1 + \sqrt{\dfrac{\left(l^2 - \eta^2\right)\left(y^2 - t^2\right)}{\left(l^2 - y^2\right)\left(\eta^2 - t^2\right)}}}{1 - \sqrt{\dfrac{\left(l^2 - \eta^2\right)\left(y^2 - t^2\right)}{\left(l^2 - y^2\right)\left(\eta^2 - t^2\right)}}}\,d\eta \tag{3.11}$$

and is of opposite sign on the left panel. Comparison of equation (3.11) with equations (3.1) and (3.4) shows that the pressure across the boundary of regions II and III is discontinuous for $c < c_1$ if $l'(c_1) \neq 0$ and is continuous for $c > c_1$.

The distinguishing feature of slender wings with cambered surfaces as compared with flat-plate wings, for either spanwise symmetric or antisymmetric loading, is that cambered surfaces carry load downstream of the position of maximum span. The pressure is discontinuous across the plane which passes through the wing maximum span unless l' is continuous and zero at $x = c_1$. Further, the pressure is continuous across the plane through the root chord unless $l'(c)$ is discontinuous.

Rolling Moment

The moment about the x-axis due to antisymmetric spanwise loading can be evaluated in a manner similar to that for the lift (ref. 5). With the asymptotic form of the complex velocity given by

$$V(x;\zeta) = -\sum_{m=1} m a_m \zeta^{-(m+1)}$$

the moment M_x about the x-axis is

$$M_x = I\left(4\pi q_\infty a_2\right)$$

The contribution to the moment in region I for the part of the wing upstream of station x, with the use of equation (1.3), is

$$M_x = -4q_\infty \int_0^l w_o(x;\eta)\sqrt{l^2 - \eta^2}\,\eta\,d\eta \tag{3.12}$$

In region II for $c > c_1$, the quantity da_2/dx is determined from the asymptotic form of equation (1.6); then,

$$\frac{dM_x}{dx} = -4q_\infty \int_0^l w_{o_x}(x;\eta)\sqrt{l^2 - \eta^2}\,\eta\,d\eta \tag{3.13}$$

and in region II for $c < c_1$, da_2/dx is evaluated from equation (3.3) and

$$\frac{dM_x}{dx} = 2\pi q_\infty\left(l^2-t^2\right)\left[Q(x)ll' - \frac{2}{\pi\left(l^2-t^2\right)}\int_t^l w_{o_x}(x;\eta)\sqrt{\left(l^2-\eta^2\right)\left(\eta^2-t^2\right)}\,d\eta\right] \tag{3.14}$$

Similarly, in region III

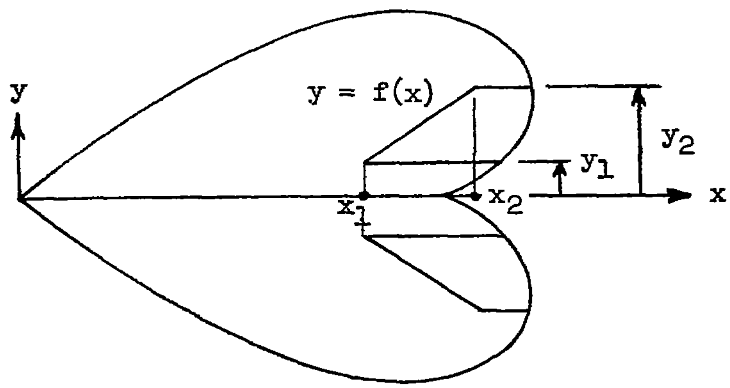
$$\frac{dM_x}{dx} = -4q_\infty\int_t^l w_{o_x}(x;\eta)\sqrt{\left(l^2 - \eta^2\right)\left(\eta^2 - t^2\right)}\,d\eta \tag{3.15}$$

The contribution to the rolling moment for the whole of region I depends only upon $w_o(c;y)$ (or $w_o\left(c_1;y\right)$ for $c > c_1$), whereas the contribution to the moment in regions II and III depends upon the distribution of camber throughout these regions.

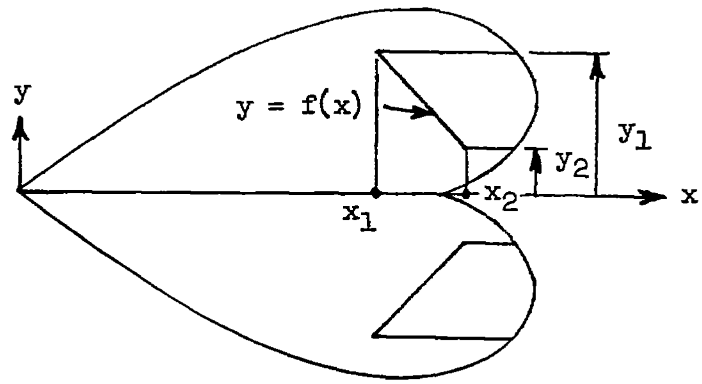
4.- CONTROL-SURFACE SOLUTIONS

The evaluation of the pressure for cambered surfaces involves considerable calculation since, in general, the various integrals must be evaluated by numerical methods. The additional load due to the deflection of a control surface, however, is readily determined from the general equations since many of the integrals can be evaluated analytically. Aside from the overall slenderness conditions required for the wing, the control surfaces should also satisfy the same conditions. Solutions are given for both symmetric and antisymmetric spanwise loading and are denoted as flaps and ailerons, respectively. These expressions apply to trailing-edge control surfaces with side edges such as those shown in sketches D and E; similar expressions for tip controls can be obtained by the same method.

The leading edge of the control is defined by $y = f(x) \equiv m_f(x - x_f)$. On the right wing panel, $f' > 0$ for the controls of sketch D and $f' < 0$ for the controls of sketch E. The coordinates of the upstream and downstream leading-edge corners are (x_1,y_1) and (x_2,y_2), respectively, and the control-surface deflection is δ_c and is positive for a downward deflection.



Sketch D



Sketch E

The additional load due to the deflection of a control surface is obtained from the general equations for cambered surfaces by setting $w_o = 0$ on the wing and $w_o = -\delta_c$ on the control surface. The quantity w_{o_x} is zero everywhere except on the edge $y = f(x)$ where w_o is discontinuous. In each of the integrals which contain w_{o_x} as a factor, the quantity w_{o_x} is interpreted as the component of a Dirac delta function, and all such integrals are of the form

$$\int w_{o_x}(x;\eta)F(x;\eta)\, d\eta = -\delta_c |f'| F(x;f)$$

where the integration across the boundary $y = f$ is in the positive y-direction; the value of the integral is zero if the integration does not extend across the edge $y = f$. The expressions for the loading apply to the configurations of both sketches D and E, with the (+) sign applicable for $f' > 0$ (sketch D) and the (-) sign applicable for $f' < 0$ (sketch E). The equations for the loading are directly applicable for x_1 in region I and x_2 in region III. The additional loading is zero upstream of x_1 for any location of x_1, and the solutions for $x > x_2$, regardless of the location of x_2, are obtained from the equations presented by replacing f by y_2 and setting $f' = 0$.

Flaps

Region I.- The pressure is zero upstream of x_1 since $w_o = 0$ on the wing, and from equation (2.1),

$$C_p = -\frac{2\delta_c}{\pi}\left[\left|f'\right| \log \frac{1 + \sqrt{\dfrac{l^2 - y^2}{l^2 - f^2}}}{1 - \sqrt{\dfrac{l^2 - y^2}{l^2 - f^2}}} \pm \frac{2ll'}{\sqrt{l^2 - y^2}}\left(\sin^{-1}\frac{f}{l} - \sin^{-1}\frac{y_1}{l}\right)\right]$$

From equation (2.22), the lifting force in region I is

$$L = \pm 4q_\infty\delta_c\left[f\sqrt{l^2 - f^2} - y_1\sqrt{l^2 - y_1{}^2} + l^2\left(\sin^{-1}\frac{f}{l} - \sin^{-1}\frac{y_1}{l}\right)\right]$$

and the total lift in region I is obtained by setting $x = c$ where $c < c_1$ and $x = c_1$ where $c > c_1$.

Region II, $c > c_1$.- From equation (2.2), the pressure coefficient in region II for $c > c_1$ is

$$C_p = -\frac{2\delta_c}{\pi}\left|f'\right| \log \frac{1 + \sqrt{\dfrac{l^2 - y^2}{l^2 - f^2}}}{1 - \sqrt{\dfrac{l^2 - y^2}{l^2 - f^2}}}$$

The lift per unit length is evaluated from equation (2.23) as

$$\frac{dL}{dx} = 8q_\infty\delta_c\left|f'\right|\sqrt{l^2 - f^2}$$

and the total lift in this region is found by integration from c_1 to c.

Region II, $c < c_1$.- The pressure coefficient for $c \leqq x \leqq c_1$ is given by equation (2.7) as

$$C_p = -2S(x)l'\left[Z(\phi,k) + \frac{y}{l}\sqrt{\frac{y^2 - t^2}{l^2 - y^2}}\right] - \frac{4\delta_c}{\pi}\left|f'\right|\Omega(x;y,f)$$

where the function $\Omega(x;y,f)$ is defined in appendix B with η replaced by f.

The function $S(x)$ is evaluated from equation (2.14). The quantity $w_o\left(l^{-1}(y);y\right)$ is zero and the left-hand member of the integral equation (2.15) is evaluated as

$$M(x) = \pm\frac{2\delta_c}{\pi}\left(f(c)\tan^{-1}\sqrt{\frac{l^2 - l^2(c)}{l^2(c) - f^2(c)}} - y_1\tan^{-1}\sqrt{\frac{l^2 - l^2(c)}{l^2(c) - y_1^2}} + \sqrt{l^2 - l^2(c)}\left[\sin^{-1}\frac{f(c)}{l(c)} - \sin^{-1}\frac{y_1}{l(c)}\right] - l\left\{\tan^{-1}\left[\frac{f(c)}{l}\sqrt{\frac{l^2 - l^2(c)}{l^2(c) - f^2(c)}}\right] - \tan^{-1}\left[\frac{y_1}{l}\sqrt{\frac{l^2 - l^2(c)}{l^2(c) - y_1^2}}\right]\right\}\right) + \int_c^x H(\xi;l(x))\,d\xi \tag{4.1}$$

where $H(\xi;l(x))$ is determined from equation (2.13) as

$$H(\xi;l(x)) = -\delta_c\left|f'(\xi)\right|\left[\frac{F(\theta,k')F(\omega,k)}{K'K} - \frac{\Gamma(\xi;f)}{K'}\right]$$

and the function $\Gamma(\xi;f)$ is given in appendix B with η replaced by f. If x_2 lies in region II, the upper limit in the integral in equation (4.1) is x_2 for $x > x_2$.

The leading terms in the function $M(x)$ are $O\left(l^2 - l^2(c)\right)^{1/2}$ whereas the contribution from the integral term is $O\left(l^2 - l^2(c)\right)^{3/2}$, and for x near c,

$$M(x) \approx \pm \frac{2\delta_c}{\pi}\left[\sin^{-1}\frac{f(c)}{l(c)} - \sin^{-1}\frac{y_1}{l(c)}\right]\sqrt{l^2 - l^2(c)} + O\left(l^2 - l^2(c)\right)^{3/2}$$

The term $O\left(l^2 - l^2(c)\right)^{3/2}$ contributes to dS/dx but not to $S(x)$ at $x = c$. Higher order terms can be retained in the approximation to $M(x)$ to improve the representation of $S(x)$. An alternate procedure which facilitates the calculations is to compute the function $M(x)$ from equation (4.1) and then represent it in the form

$$M(x) = \frac{2\delta_c}{\pi}\sqrt{l^2-l^2(c)}\left\{\pm\left[\sin^{-1}\frac{f(c)}{l(c)} - \sin^{-1}\frac{y_1}{l(c)}\right] + \sum_{n=1}^{m}\frac{a_n}{l^{2n}(c)}\left[l^2-l^2(c)\right]^{2n}\right\} \tag{4.2}$$

With this representation the function $S(x)$ is evaluated from equation (2.16) in the neighborhood of $x = c$ as

$$S(x) = \frac{2\delta_c}{\pi k}\left\{\pm\left[\sin^{-1}\frac{f(c)}{l(c)} - \sin^{-1}\frac{y_1}{l(c)}\right]\left(1 + \frac{4}{\gamma}\int_0^{\frac{k'}{4}}\frac{d\mu}{\log \mu}\right) + \sum_{n=1}^{m}\frac{a_n}{2^{2n}}\frac{(2n+1)!}{(n!)^2}\left[\frac{l^2 - l^2(c)}{l^2(c)}\right]^n\right\} \tag{4.3}$$

and at $x = c$,

$$S(c) = \pm\frac{2\delta_c}{\pi}\left[\sin^{-1}\frac{f(c)}{l(c)} - \sin^{-1}\frac{y_1}{l(c)}\right]$$

The function $S(x)$ can be evaluated by the method of appendix C in regions where this approximation is not suitable.

The lift per unit length is determined from equation (2.24) as

$$\frac{dL}{dx} = 4\pi q_{\infty} l\left[S(x) l'\left(1 - \frac{E}{K}\right) + \frac{2\delta_c}{\pi}\left|f'\right| Z(\omega,k)\right]$$

where $\sin \omega = \sqrt{\frac{l^2 - f^2}{l^2 - t^2}}$, and the total lift in region II for $c < c_1$ is obtained by integration from c to c_1.

Region III.- The pressure coefficient in region III for $x < x_2$ is evaluated from equation (2.20) as

$$C_p = -\frac{4\delta_c}{\pi}\left|f'\right| \Omega(x;y,f)$$

and is zero for $x > x_2$. The lift per unit length is obtained from equation (2.25) as

$$\frac{dL}{dx} = 8q_{\infty}\delta_c l\left|f'\right| Z(\omega,k)$$

with η replaced by f in the definition of ω, and the total lift over region III is obtained by integration from c_1 or c, whichever is larger, to x_2.

From the foregoing equations and the modifications for $x > x_2$, it can be seen that the additional loading due to the deflection of a flap is different from zero only for $x_1 < x < x_2$ or c_1, whichever is larger, regardless of the location of x_1 and x_2. Further, the pressure and lift distributions are discontinuous at x_1 and x_2 and at c_1 for $x_2 > c_1$ if $l'(c_1) \neq 0$.

Ailerons

The equations for the additional loading due to the deflection of an aileron are developed in the same manner as those for the flap except that the basic equations are those for antisymmetric spanwise loading. The solutions for the aileron are somewhat simpler, however, since they involve only elementary functions. The equations presented apply directly to the configurations of sketches D and E with x_1 in region I and x_2 in region III. The additional loading is zero upstream of x_1 for any location of x_1, and the solutions for $x > x_2$ for any location of x_2 are obtained from the equations presented by replacing f by y_2 and setting $f' = 0$.

Region I.- The pressure coefficient is zero upstream of x_1, and for $x > x_1$, the pressure coefficient on the right wing panel is evaluated from equation (3.1) as

$$C_p = -\frac{2\delta_c}{\pi}\left[|f'|\log\left|\frac{1 + \frac{f}{y}\sqrt{\frac{l^2 - y^2}{l^2 - f^2}}}{1 - \frac{f}{y}\sqrt{\frac{l^2 - y^2}{l^2 - f^2}}}\right| \pm \frac{2yl'}{l\sqrt{l^2 - y^2}}\left(\sqrt{l^2 - y_1{}^2} - \sqrt{l^2 - f^2}\right)\right]$$

From equation (3.12), the rolling moment is

$$M_x = \pm\frac{4}{3}q_\infty\delta_c\left[\left(l^2 - y_1{}^2\right)^{3/2} - \left(l^2 - f^2\right)^{3/2}\right]$$

and the total moment from region I is obtained by setting the argument of l and f equal to the smaller of c or c_1.

Region II, $c > c_1$.- The pressure coefficient in region II for $c > c_1$ on the right panel is evaluated from equation (3.4) as

$$C_p = -\frac{2\delta_c}{\pi}\left|f'\right|\log\left|\frac{1 + \frac{f}{y}\sqrt{\frac{l^2 - y^2}{l^2 - f^2}}}{1 - \frac{f}{y}\sqrt{\frac{l^2 - y^2}{l^2 - f^2}}}\right|$$

From equation (3.13), the rolling moment per unit length is

$$\frac{dM_x}{dx} = 4q_\infty\delta_c\left|f'\right|f\sqrt{l^2 - f^2}$$

and the total contribution to the moment is obtained by integration on x from c_1 to c.

Region II, $c < c_1$.- From equation (3.4), the pressure coefficient for $t \leqq y \leqq l$ is

$$C_p = -\frac{2\delta_c}{\pi}\left|f'\right|\log\frac{1 + \sqrt{\frac{\left(l^2 - f^2\right)\left(y^2 - t^2\right)}{\left(l^2 - y^2\right)\left(f^2 - t^2\right)}}}{1 - \sqrt{\frac{\left(l^2 - f^2\right)\left(y^2 - t^2\right)}{\left(l^2 - y^2\right)\left(f^2 - t^2\right)}}} - 2Q(x)ll'\sqrt{\frac{y^2 - t^2}{l^2 - y^2}}$$

where $Q(x)$ is determined from the integral equation (3.6). The quantity $w_o\left(l^{-1}(y);y\right)$ is zero, and from equation (3.7), the left-hand member of the integral equation is

$$N(x) = \pm \frac{2\delta_c}{\pi}\left\{f(c)\tan^{-1}\left[\frac{f(c)}{l}\sqrt{\frac{l^2-l^2(c)}{l^2(c)-f^2(c)}}\right] - y_1\tan^{-1}\left[\frac{y_1}{l}\sqrt{\frac{l^2-l^2(c)}{l^2(c)-{y_1}^2}}\right] - \right.$$

$$\left. l\left[\tan^{-1}\sqrt{\frac{l^2-l^2(c)}{l^2(c)-f^2(c)}} - \tan^{-1}\sqrt{\frac{l^2-l^2(c)}{l^2(c)-{y_1}^2}}\right]\right\} + \int_c^x h(\xi;l(x))\,d\xi \tag{4.4}$$

where $h(\xi;l(x))$ is evaluated from equation (3.8) as

$$h(\xi;l(x)) = \frac{2\delta_c}{\pi}|f'|\tan^{-1}\sqrt{\frac{\left[l^2(x) - l^2\right]\left(f^2 - t^2\right)}{\left[l^2(x) - t^2\right]\left(l^2 - f^2\right)}}$$

If x_2 lies in region II for $c < c_1$, the upper limit in equation (4.4) is x_2 for $x > x_2$. The dominant terms in equation (4.4) are $O\left(l^2 - l^2(c)\right)^{1/2}$ for x near c and the contribution from the integral is $O\left(l^2 - l^2(c)\right)^{3/2}$; only the terms of $O\left(l^2 - l^2(c)\right)^{1/2}$ contribute to $Q(c)$ but the higher order terms contribute to dQ/dx at $x = c$.

The function $Q(x)$ for x near c is conveniently evaluated by computing $N(x)$ from equation (4.4) and then representing it in the form

$$N(x) = \frac{2\delta_c}{\pi}\sqrt{l^2-l^2(c)}\left(\pm\left\{\frac{1}{l}\left[\frac{f^2(c)}{\sqrt{l^2(c)-f^2(c)}} - \frac{{y_1}^2}{\sqrt{l^2(c)-{y_1}^2}}\right] - l\left[\frac{1}{\sqrt{l^2(c)-f^2(c)}} - \frac{1}{\sqrt{l^2(c)-{y_1}^2}}\right]\right\} + \right.$$

$$\left.\sum_{n=1}^{m}\frac{a_n}{l^{2n+1}(c)}\,l\left[l^2-l^2(c)\right]^n\right) \tag{4.5}$$

The quantity in braces is the term $O\left(l^2 - l^2(c)\right)^{1/2}$ obtained by expanding equation (4.4). From equation (3.9) then,

$$Q(x)l = \frac{2\delta_c}{\pi}\left(\pm\left\{\frac{l(c)}{l^2}\left[\frac{f^2(c)}{\sqrt{l^2(c)-f^2(c)}} - \frac{{y_1}^2}{\sqrt{l^2(c)-{y_1}^2}}\right] - l\left[\frac{1}{\sqrt{l^2(c)-f^2(c)}} - \frac{1}{\sqrt{l^2(c)-{y_1}^2}}\right]\right\} + \sum_{n=1}^{m} a_n \frac{(2n+1)!}{2^{2n}(n!)^2}\frac{l}{l(c)}\left[\frac{l^2-l^2(c)}{l^2(c)}\right]^n\right) \tag{4.6}$$

and

$$Q(c)l(c) = \pm\frac{2\delta_c}{\pi l(c)}\left[\sqrt{l^2(c) - {y_1}^2} - \sqrt{l^2(c) - f^2(c)}\right]$$

The function $Q(x)$ can be evaluated by the method of appendix C in regions where equation (4.6) does not provide a suitable approximation.

The moment per unit length is evaluated from equation (3.14) as

$$\frac{dM_x}{dx} = 2\pi q_\infty\left(l^2 - t^2\right)\left[Q(x)ll' + \frac{2\delta_c}{\pi}\frac{|f'|}{l^2 - t^2}\sqrt{\left(l^2 - f^2\right)\left(f^2 - t^2\right)}\right] \tag{4.7}$$

and integration on x from c to c_1 gives the contribution to the moment from the part of the wing in region II.

Region III.- From equation (3.11), the pressure coefficient in region III for $t \leqq y \leqq l$ is

$$C_p = -\frac{2\delta_c}{\pi}\left|f'\right| \log \frac{1 + \sqrt{\frac{\left(l^2 - f^2\right)\left(y^2 - t^2\right)}{\left(l^2 - y^2\right)\left(f^2 - t^2\right)}}}{1 - \sqrt{\frac{\left(l^2 - f^2\right)\left(y^2 - t^2\right)}{\left(l^2 - y^2\right)\left(f^2 - t^2\right)}}}$$

and is zero for $x > x_2$. Evaluating the integral in equation (3.15) gives the rolling moment per unit length as

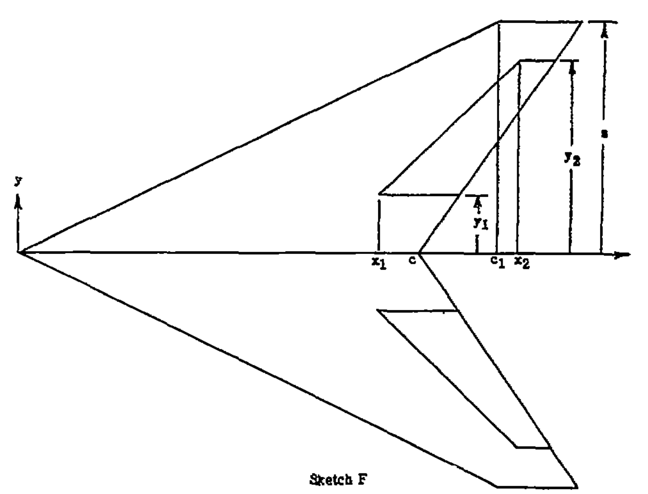
$$\frac{dM_x}{dx} = 4q_\infty\delta_c\left|f'\right| \sqrt{\left(l^2 - f^2\right)\left(f^2 - t^2\right)}$$

and the moment from the part of the wing in region III is found by integration on x from c or c_1, whichever is larger, to x_2.

As in the case of the loading due to a flap, the additional loading due to the deflection of an aileron is different from zero only for $x_1 < x < x_2$ or c_1, whichever is larger, regardless of the location of x_1. Similarly, the pressure and lift distributions are discontinuous at x_1 and x_2 and at c_1 for $x_2 > c_1$ if $l'(c_1) \neq 0$.

Example.- Calculations have been made for the swept-wing configuration with an aileron shown in sketch F. The leading and trailing edges of the right wing panel are defined by $l = m_l x$ and $t = m_t(x - c)$, respectively, and the relevant constants are

$m_l = 0.5$ $m_t = 1.5$
$m_f = 1$ $c_1/c = 1.2$
$x_1/c = 0.9$ $x_2/c = 1.25$
$x_f/c = 0.75$ $s/c = 0.6$
$y_1/c = 0.15$ $y_2/c = 0.5$



Sketch F

The quantity $\frac{Qll'}{\delta_c}$ calculated by the numerical method of appendix C is compared in figure 2 with that calculated from equation (4.6) with $m = 3$. An interval of 0.02 was used in the numerical method. The coefficients were evaluated so that equation (4.5) gave the correct value of $N(x)$ at $x = 1.02$, 1.06, and 1.20. The results show that the approximate equation for $Q(x)$ provides a very satisfactory solution even when t/l is not small compared to unity. For example, at $x/c = 1.14$, the value of t/l is 0.37 and the error is approximately 5 percent. In figure 3 the rolling-moment distribution is shown as a function of x/c. The moment distribution is zero for $x/c < 0.90$ and $x/c > 1.25$; these values correspond to the downstream and upstream extremities of the aileron leading edge. Further, the moment distribution is discontinuous at the boundary of regions II and III. The chordwise pressure distributions for $\frac{y}{l(c)} = 0.2$, 0.6, and 1.1 are shown in figure 4.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., January 31, 1958.

APPENDIX A

SYMBOLS

c	wing root chord
c_1	distance from apex to position of maximum span of wing
s	wing semispan
x,y,z	Cartesian coordinates (x is in the stream direction)
ξ,η,μ	dummy variables of integration
$\zeta = y + iz$	
$l(x)$	spanwise ordinate of wing leading edge
$l'(x) = \frac{dl}{dx}$	
$l^{-1}(y)$	chordwise ordinate of wing leading edge
$f(x)$	spanwise ordinate of control-surface leading edge, $f(x) \equiv m_f(x - x_f)$
$f'(x) = \frac{df}{dx}$	
$t(x)$	spanwise ordinate of wing trailing edge
$\gamma = \left.\frac{dt}{dl}\right\|_{x=c}$	
(x_1,y_1)	upstream coordinates of control-surface leading-edge corner
(x_2,y_2)	downstream coordinates of control-surface leading-edge corner
δ_c	control-surface deflection
u,v,w	nondimensional disturbance velocities in x-, y-, and z-directions, respectively
φ	disturbance velocity potential

$v = \int w_o(x;y)dy$

$V(x;\zeta) = v - iw$

L	lift
M_x	moment about x-axis
C_p	pressure coefficient on the wing
$P(x;\zeta)$	polynomial in ζ
A,B	coefficients of polynomial
q_∞	stream dynamic pressure
a_m	defined by equation (2.21)
$G(x;y)$	defined by equation (2.5)
$g(\xi;l(x))$	defined by equation (2.11)
$H(\xi;l(x))$	defined by equation (2.13)
$h(\xi;l(x))$	defined by equation (3.8)
$S(x)$	evaluated from equation (2.14)
$M(x)$	defined by equation (2.15)
$Q(x)$	evaluated from equation (3.6)
$N(x)$	defined by equation (3.7)
K,E	complete elliptic integrals of first and second kind, respectively, with modulus k
K',E'	complete elliptic integrals of first and second kind, respectively, with modulus k'
$F(\theta,k),E(\theta,k)$	incomplete elliptic integrals of first and second kind, respectively
$\Pi(\theta,\alpha^2,k)$	elliptic integral of third kind
$Z(\phi,k)$	Jacobian zeta function

$\Lambda_o(\theta,k)$ Heuman's lambda function

$$k = \sqrt{1 - \frac{t^2}{l^2}}$$

$$k' = \frac{t}{l}$$

$$\alpha^2 = \frac{l^2 - t^2}{l^2 - \eta^2}$$

$$\alpha'^2 = 1 - \alpha^2$$

$$\omega = \sin^{-1} \frac{1}{\alpha}$$

$$\phi = \sin^{-1} \sqrt{\frac{l^2 - y^2}{l^2 - t^2}}$$

$$\theta = \sin^{-1} \sqrt{\frac{l^2(x) - l^2}{l^2(x) - t^2}}$$

I imaginary part

$\Omega(x;y,\eta)$ defined by equations (B3)

$\Gamma(\xi;\eta)$ defined by equation (B9)

$$\sigma = \frac{\pi}{2} \frac{F(\omega,k)}{K}$$

$$\sigma' = \frac{\pi}{2} \frac{F(\omega,k)}{K'}$$

$$\tau = \frac{\pi}{2} \frac{F(\phi,k)}{K}$$

$$\tau' = \frac{\pi}{2} \frac{F(\theta,k')}{K'}$$

$$p = \frac{\pi}{2} \frac{K'}{K}$$

$p' = \frac{\pi}{2} \frac{K}{K'}$

$q = e^{-2p}$

$q' = e^{-2p'}$

Subscripts:

o	values on $z = +0$
x	partial derivative with respect to x
y	partial derivative with respect to y

APPENDIX B

EVALUATION OF $G(x;y)$, $g(\xi;l(x))$, AND RELATED FUNCTIONS

The solutions for the slender wing with symmetric spanwise loading involve elliptic integrals of the third kind in region II for $c < c_1$ and in region III. These integrals which arise in the evaluation of the functions $G(x;y)$ and $g(\xi;l(x))$ and in the expression for the pressure in region III are reduced to a form suitable for computation by their relation to theta functions. The functions $\Omega(x;y,\eta)$ and $\Gamma(\xi;\eta)$, which are logarithms of theta functions, are readily evaluated in series form. These relations, as well as the reduction of the elliptic integrals which arise in the solution for symmetric spanwise loading, are given in reference 8, for example.

The Functions Related to the Pressure

With an interchange of the order of integration and with the use of partial fractions, the function $G(x;y)$ given by equation (2.5) can be evaluated as

$$G(x;y) = -\frac{2}{\pi}\int_{t}^{l} w_{o_x}(x;\eta)\,\frac{\eta}{l}\sqrt{\frac{\eta^2-t^2}{l^2-\eta^2}}\left\{\frac{y}{l}\,\frac{l^2-\eta^2}{l^2-t^2}\sqrt{\frac{y^2-t^2}{l^2-y^2}}+\frac{l^2-\eta^2}{l^2-t^2}\Big[E(\phi,k)-F(\phi,k)\Big]-\Pi(\phi,\alpha^2,k)\right\}d\eta \qquad \text{(B1)}$$

where $F(\phi,k)$, $E(\phi,k)$, and $\Pi(\phi,\alpha^2,k)$ are elliptic integrals of the first, second, and third kind, respectively, and

$$\sin\phi = \sqrt{\frac{l^2-y^2}{l^2-t^2}} \qquad k^2 = 1-\frac{t^2}{l^2} \qquad \alpha^2 = \frac{l^2-t^2}{l^2-\eta^2}$$

For $y = t$, the argument ϕ is $\frac{\pi}{2}$, and the elliptic integrals are complete.

This elliptic integral of the third kind in equation (B1) can be evaluated in terms of the Jacobian zeta function and a theta function. For $1 < \alpha^2 < \infty$,

$$
\left.
\begin{aligned}
\Pi(\phi,\alpha^2,k) &= -\frac{l}{\eta}\sqrt{\frac{l^2 - \eta^2}{\eta^2 - t^2}}\left[F(\phi,k)Z(\omega,k) - \Omega(x;\eta)\right] \\
\Pi\left(\frac{\pi}{2},\alpha,k\right) &= -\frac{l}{\eta}\sqrt{\frac{l^2 - \eta^2}{\eta^2 - t^2}}\,K\,Z(\omega,k)
\end{aligned}
\right\} \qquad \text{(B2)}
$$

where

$$
\left.
\begin{aligned}
\Omega(x;y,\eta) &= \frac{1}{2}\log\frac{\sin(\sigma + \tau)}{\sin(\sigma - \tau)} + \sum_{m=1}^{\infty} q^m \frac{\sin(2m\sigma)\sin(2m\tau)}{m\sinh(2mp)} \\
\Omega(x;y,\eta) &= \frac{1}{2}\log\frac{1 + \sqrt{\dfrac{l^2 - y^2}{l^2 - \eta^2}}}{1 - \sqrt{\dfrac{l^2 - y^2}{l^2 - \eta^2}}} \qquad (\text{for } x = c)
\end{aligned}
\right\} \qquad \text{(B3)}
$$

and

$$
\sin\omega = \frac{1}{\alpha} = \sqrt{\frac{l^2 - \eta^2}{l^2 - t^2}} \qquad \sigma = \frac{\pi}{2}\frac{F(\omega,k)}{K} \qquad \tau = \frac{\pi}{2}\frac{F(\phi,k)}{K}
$$

$$
p = \frac{\pi}{2}\frac{K'}{K} \qquad q = e^{-2p}
$$

From the definition of the zeta function $Z(\phi,k) = E(\phi,k) - \frac{E}{K}F(\phi,k)$ and equations (B2), the combination of the functions $G(x;y)$ and $G(x;t)$ which arises in the expression for the pressure is

$$\left.\begin{aligned}
&G(x;y) - \frac{F(\phi,k)}{K}\,G(x;t) = -\frac{2}{\pi l\left(l^2-t^2\right)}\left[\frac{y}{l}\sqrt{\frac{y^2-t^2}{l^2-y^2}} + Z(\phi,k)\right]\int_t^l w_{o_x}(x;\eta)\sqrt{\left(l^2-\eta^2\right)\left(\eta^2-t^2\right)}\,\eta\,d\eta + \frac{2}{\pi}\int_t^l w_{o_x}(x;\eta)\Omega(x;\eta)d\eta \qquad (y > t)\\
&G(x;y) - \frac{F(\phi,k)}{K}\,G(x;t) = 0 \qquad (y = t)
\end{aligned}\right\} \tag{B4}$$

For x near c, the function $\Omega(x;\eta)$ can be approximated by the first term of equations (B3).

The Functions Related to the Integral Equation

With an interchange of the order of integration and with the use of partial fractions in equation (2.11), the integral on y can be expressed as the sum of two integrals, and the function $g(\xi;l(x))$ is evaluated as

$$g(\xi;l(x)) = -\frac{2}{\pi}\int_t^l w_{o_\xi}(\xi;\eta)\,\frac{\eta}{l}\sqrt{\frac{l^2-\eta^2}{\eta^2-t^2}}\left\{\frac{l(x)}{l}\,\frac{\eta^2-t^2}{\sqrt{\left[l^2(x)-l^2\right]\left[l^2(x)-t^2\right]}} - F(\theta,k') + \frac{\eta^2-t^2}{l^2-t^2}E(\theta,k') + \frac{l^2-t^2}{l^2-\eta^2}\Pi(\theta,\alpha'^2,k')\right\}d\eta \tag{B5}$$

where

$$\sin\theta = \sqrt{\frac{l^2(x) - l^2}{l^2(x) - t^2}} \qquad k' = \frac{t}{l} \qquad \alpha'^2 = 1 - \alpha^2 = -\frac{\eta^2 - t^2}{l^2 - \eta^2}$$

From equations (B1) and (B5), the quantity $g(\xi;l(x)) + \frac{F(\theta,k')}{K}G(\xi;t)$ which arises in the integral equation for $S(x)$ can be expressed as

$$g(\xi;l(x)) + \frac{F(\theta,k')}{K} G(\xi;t) = -\frac{2}{\pi l\left(l^2-t^2\right)}\left\{\frac{l\, l(x)k^2}{\sqrt{\left[l^2(x)-l^2\right]\left[l^2(x)-t^2\right]}} + \frac{\pi}{2}\frac{\Lambda_0(\theta,k)}{K}\right\}\int_t^l w_{o_\xi}(\xi;\eta)\sqrt{\left(l^2-\eta^2\right)\left(\eta^2-t^2\right)}\,\eta\, d\eta + H(\xi;l(x)) \tag{B6}$$

where

$$H(\xi;l(x)) = -\frac{2}{\pi}\int_t^l w_{o_\xi}(\xi;\eta)\,\frac{\eta}{l}\sqrt{\frac{l^2-\eta^2}{\eta^2-t^2}}\left[\frac{l^2-t^2}{l^2-\eta^2}\Pi(\theta,\alpha'^2,k') - F(\theta,k') - \frac{\eta^2-t^2}{l^2-\eta^2}\frac{F(\theta,k')}{K}\Pi\left(\frac{\pi}{2},\alpha^2,k\right)\right]d\eta \tag{B7}$$

and Heuman's lambda function (ref. 10) is defined by the relation

$$\Lambda_0(\theta,k) = \frac{2}{\pi}\left[E\ F(\theta,k') + K\ E(\theta,k') - K\ F(\theta,k')\right]$$

The parameter α'^2 is zero for $\eta = t$; then, $\Pi(\theta,0,k') = F(\theta,k')$ and the function $H(\xi;l(x))$ is zero. For $\eta > t$, the elliptic integral $\Pi(\theta,\alpha'^2,k')$ can be expressed in terms of Heuman's lambda function and a theta function as

$$\frac{l^2-t^2}{l^2-\eta^2}\Pi(\theta,\alpha'^2,k') = F(\theta,k') - \frac{\pi}{2K'}\frac{l}{\eta}\sqrt{\frac{\eta^2-t^2}{l^2-\eta^2}}\left[F(\theta,k')\Lambda_0(\omega,k') - \Gamma(\xi;\eta)\right] \tag{B8}$$

where

$$\Gamma(\xi;\eta) = F(\theta,k') + \tan^{-1}\left\{\frac{2\sum_{m=1}^{\infty}(-1)^{m+1}q'^{m^2}\sin(2m\tau')\sinh\left[2m(p'-\sigma')\right]}{1+2\sum_{m=1}^{\infty}(-1)^{m}q'^{m^2}\cos(2m\tau')\cosh\left[2m(p'-\sigma')\right]}\right\} \tag{B9}$$

and

$$\sigma' = \frac{\pi}{2}\frac{F(\omega,k)}{K'} \qquad \tau' = \frac{\pi}{2}\frac{F(\theta,k')}{K'} \qquad p' = \frac{\pi}{2}\frac{K}{K'} \qquad q' = e^{-2p'}$$

Substituting from equations (B2) and (B8) into equation (B7), making use of the definitions of $\Lambda_o(\omega,k')$ and $Z(\omega,k)$, and using Legendre's relation $EK' + E'K - KK' = \frac{\pi}{2}$ give the function $H(\xi;l(x))$ as

$$H(\xi;l(x)) = \int_t^l w_{o_\xi}(\xi;\eta)\left[\frac{F(\theta,k')F(\omega,k)}{K'K} - \frac{\Gamma(\xi;\eta)}{K'}\right]d\eta \tag{B10}$$

For small values of θ, the functions $F(\theta,k')$ and $\Pi(\theta,\alpha'^2, k')$ are approximated by $\sin\theta = \sqrt{\frac{l^2(x) - l^2}{l^2(x) - t^2}}$, and from equations (B7) and (B2), the function $H(\xi;l(x))$ for x near c becomes

$$H(\xi;l(x)) = -\frac{2}{\pi}\sqrt{\frac{l^2(x) - l^2}{l^2(x) - t^2}}\int_t^l w_{o_\xi}(\xi;\eta)\left[\frac{\eta}{l}\sqrt{\frac{\eta^2 - t^2}{l^2 - \eta^2}} + Z(\omega,k)\right]d\eta \tag{B11}$$

An alternate form for x near c is obtained by making use of the approximation $\Pi(\theta,\alpha'^2,k') = \frac{1}{\alpha}\tan^{-1}(\alpha\tan\theta) + O(k'^2)$. The function $H(\xi;l(x))$ then becomes

$$H(\xi;l(x)) = -\frac{2}{\pi}\int_t^l w_{o_\xi}(\xi;\eta)\left\{\frac{\eta}{l}\sqrt{\frac{l^2-\eta^2}{\eta^2-t^2}}\left[\sqrt{\frac{l^2-t^2}{l^2-\eta^2}}\tan^{-1}\sqrt{\frac{l^2(x)-l^2}{l^2-\eta^2}} - \sqrt{\frac{l^2(x)-l^2}{l^2(x)-t^2}}\right] + \right.$$

$$\left.\sqrt{\frac{l^2(x)-l^2}{l^2(x)-t^2}}Z(\omega,k)\right\}d\eta$$

APPENDIX C

NUMERICAL SOLUTION OF THE INTEGRAL EQUATIONS

FOR THE FUNCTIONS $S(x)$ AND $Q(x)$

The kernels of the integral equations for the functions $S(x)$ and $Q(x)$, which arise in the solution for the pressures in region II for $c < c_1$, are equivalent to those given by Mirels for the flat-plate wing at angle of attack and for roll. The integral equations for the cambered wings differ from those for the flat plate only in the left-hand members. Since these equations can be inverted analytically in the neighborhood of $x = c$, the inversion in general can only be accomplished by numerical methods. The numerical method of solution presented in this appendix is based upon the method developed in reference 5.

Solution for $S(x)$

Let the range of integration be divided into n intervals of length Δ and let the end points of the intervals be denoted by $x_i = c + i\,\Delta$ and the midpoints of the intervals by $\bar{x}_i = c + \left(i - \frac{1}{2}\right)\Delta$ where $i = 0, 1, \ldots n$ and $x = x_n$. The integral equation for $S(x)$ given by equation (2.14) can then be expressed as

$$M(x) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} S(\xi) l' \left\{ \frac{\pi}{2} \frac{\Lambda_0(\theta,k)}{K} + \frac{k^2 l\ l(x)}{\sqrt{\left[l^2(x) - l^2\right]\left[l^2(x) - t^2\right]}} \right\} d\xi$$

With the notation

$$\Psi(x;\xi) = \frac{\pi}{2} \frac{\Lambda_0(\theta,k)}{K} l' \qquad l_i = l(x_i) \qquad \bar{l}_i = l(\bar{x}_i)$$

$$\bar{t}_i = t(\bar{x}_i) \qquad \bar{k}_i^{\,2} = 1 - \frac{\bar{t}_i^{\,2}}{\bar{l}_i^{\,2}}$$

and with the use of the mean-value theorem, the integral equation is approximated as

$$M(x_n) = \sum_{i=1}^{n} S(\bar{x}_i) \left[\int_{x_{i-1}}^{x_i} \Psi(x_n;\xi)d\xi + \bar{k}_i^{\,2} l_n \frac{\sqrt{l_n^2 - l_{i-1}^2} - \sqrt{l_n^2 - l_i^2}}{\sqrt{l_n^2 - \bar{t}_i^{\,2}}}\right]$$

Solving for $S(\bar{x}_n)$ gives

$$S(\bar{x}_n) = \frac{M(x_n) - \sum_{i=1}^{n-1} S(\bar{x}_i)\left[\int_{x_{i-1}}^{x_i} \Psi(x_n;\xi)d\xi + \bar{k}_i^{\,2} l_n \frac{\sqrt{l_n^2 - l_{i-1}^2} - \sqrt{l_n^2 - l_i^2}}{\sqrt{l_n^2 - \bar{t}_i^{\,2}}}\right]}{\int_{x_{n-1}}^{x_n} \Psi(x_n;\xi)d\xi + \bar{k}_n^{\,2} l_n \frac{\sqrt{l_n^2 - l_{n-1}^2}}{\sqrt{l_n^2 - \bar{t}_n^{\,2}}}} \tag{C1}$$

The values of $S(\bar{x}_1)$, $S(\bar{x}_2)$, . . . are found by successively setting $n = 1, 2, \ldots$. The function $\Psi(x_n;\xi)$ is regular everywhere and vanishes for $\xi = c$ and $\xi = x_n$, and the integrals can be evaluated by Simpson's rule or other suitable methods.

Solution for $Q(x)$

The integral equation (3.6) can be solved by the same method as that given for the evaluation of $S(x)$. Dividing the range of integration c to x into n intervals gives equation (3.6) as

$$N(x) = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} Q(\xi) l l' \sqrt{\frac{l^2(x) - t^2}{l^2(x) - l^2}}\, d\xi$$

and with the use of the mean-value theorem, $Q(\bar{x}_n)$ is determined as

$$Q(\bar{x}_n) = \frac{N(x_n) - \sum_{i=1}^{n-1} Q(\bar{x}_i)\sqrt{l_n^2 - \bar{t}_i^{\,2}}\left(\sqrt{l_n^2 - l_{i-1}^2} - \sqrt{l_n^2 - l_i^2}\right)}{\sqrt{l_n^2 - \bar{t}_n^{\,2}}\sqrt{l_n^2 - l_{n-1}^2}} \tag{C2}$$

The values $Q(\bar{x}_1)$, $Q(\bar{x}_2)$, . . . are obtained by successively setting $n = 1, 2, \ldots$.

REFERENCES

1. Jones, Robert T.: Properties of Low-Aspect-Ratio Pointed Wings at Speeds Below and Above the Speed of Sound. NACA Rep. 835, 1946. (Supersedes NACA TN 1032.)

2. Jones, Robert T., and Cohen, Doris: Aerodynamics of Wings at High Speeds. Aerodynamic Components of Aircraft at High Speeds. Vol. VII of High Speed Aerodynamics and Jet Propulsion, sec. A, A. F. Donovan and H. R. Lawrence, eds., Princeton Univ. Press, 1957, pp. 3-243.

3. Heaslet, Max. A., and Lomax, Harvard: Supersonic and Transonic Small Perturbation Theory. General Theory of High Speed Aerodynamics. Vol. VI of High Speed Aerodynamics and Jet Propulsion, sec. D, W. R. Sears, ed., Princeton Univ. Press, 1954, pp. 122-344.

4. Lomax, Harvard, and Heaslet, Max. A.: Linearized Lifting-Surface Theory for Swept-Back Wings With Slender Plan Forms. NACA TN 1992, 1949.

5. Mirels, Harold: Aerodynamics of Slender Wings and Wing-Body Combinations Having Swept Trailing Edges. NACA TN 3105, 1954.

6. Mangler, K. W.: Calculation of the Pressure Distribution Over a Wing at Sonic Speeds. R. & M. No. 2888, British A.R.C., Sept. 1951.

7. Cheng, H. K., and Rott, N.: Generalizations of the Inversion Formula of Thin Airfoil Theory. Jour. Rational Mech. and Analysis, vol. 3, no. 3, 1954, pp. 357-382.

8. Byrd, Paul F., and Friedman, Morris D.: Handbook of Elliptic Integrals for Engineers and Physicists. Lange, Maxwell & Springer, Ltd. (New York), 1954.

9. Milne-Thomson, L. M.: Jacobian Elliptic Function Tables. Dover Publications, Inc., 1950.

10. Heuman, Carl: Tables of Complete Elliptic Integrals. Jour. Math. and Phys., vol. XX, no. 2, Apr. 1941, pp. 127-206; Corrections in vol. XX, no. 3, Aug. 1941, p. 336.

11. Jahnke, Eugene, and Emde, Fritz: Tables of Functions. Fourth ed., Dover Publications, 1945.

12. Ward, G. N.: Linearized Theory of Steady High-Speed Flow. Cambridge Univ. Press, 1955.

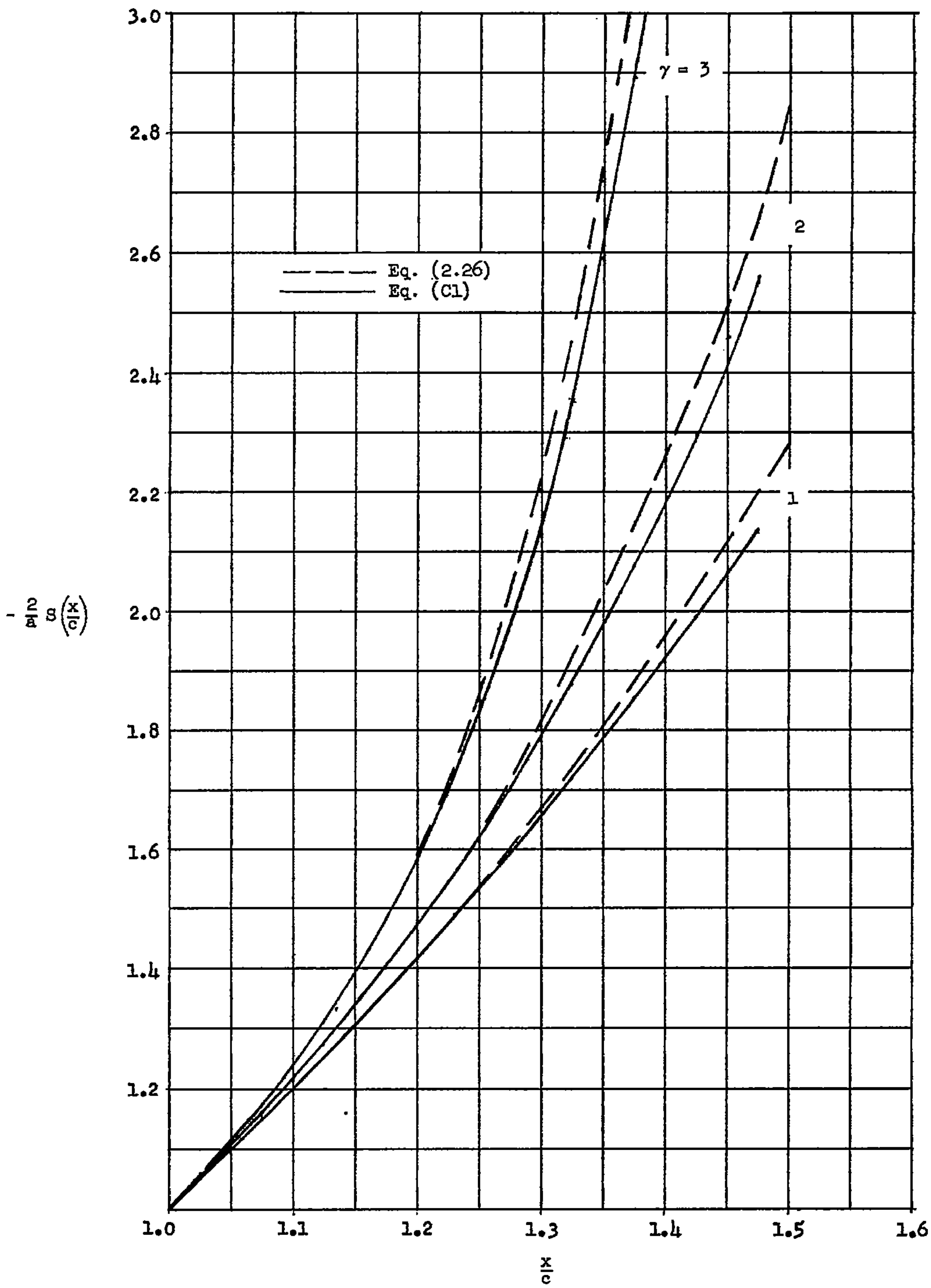


Figure 1.- The function $S\left(\frac{x}{c}\right)$ for arrow wing with quadratic twist.

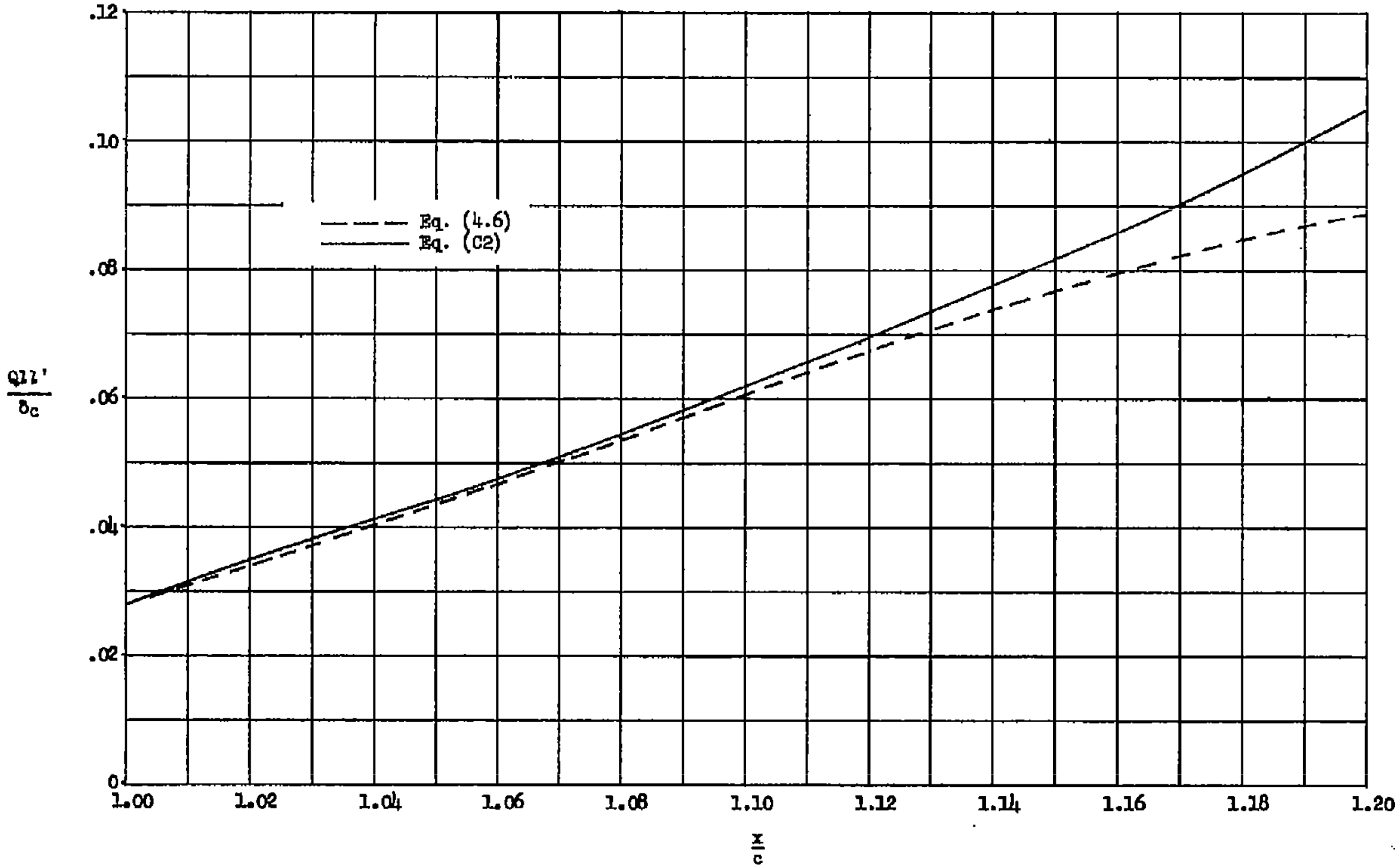


Figure 2.- The function $\frac{Qll'}{\delta_c}$ for wing with aileron.

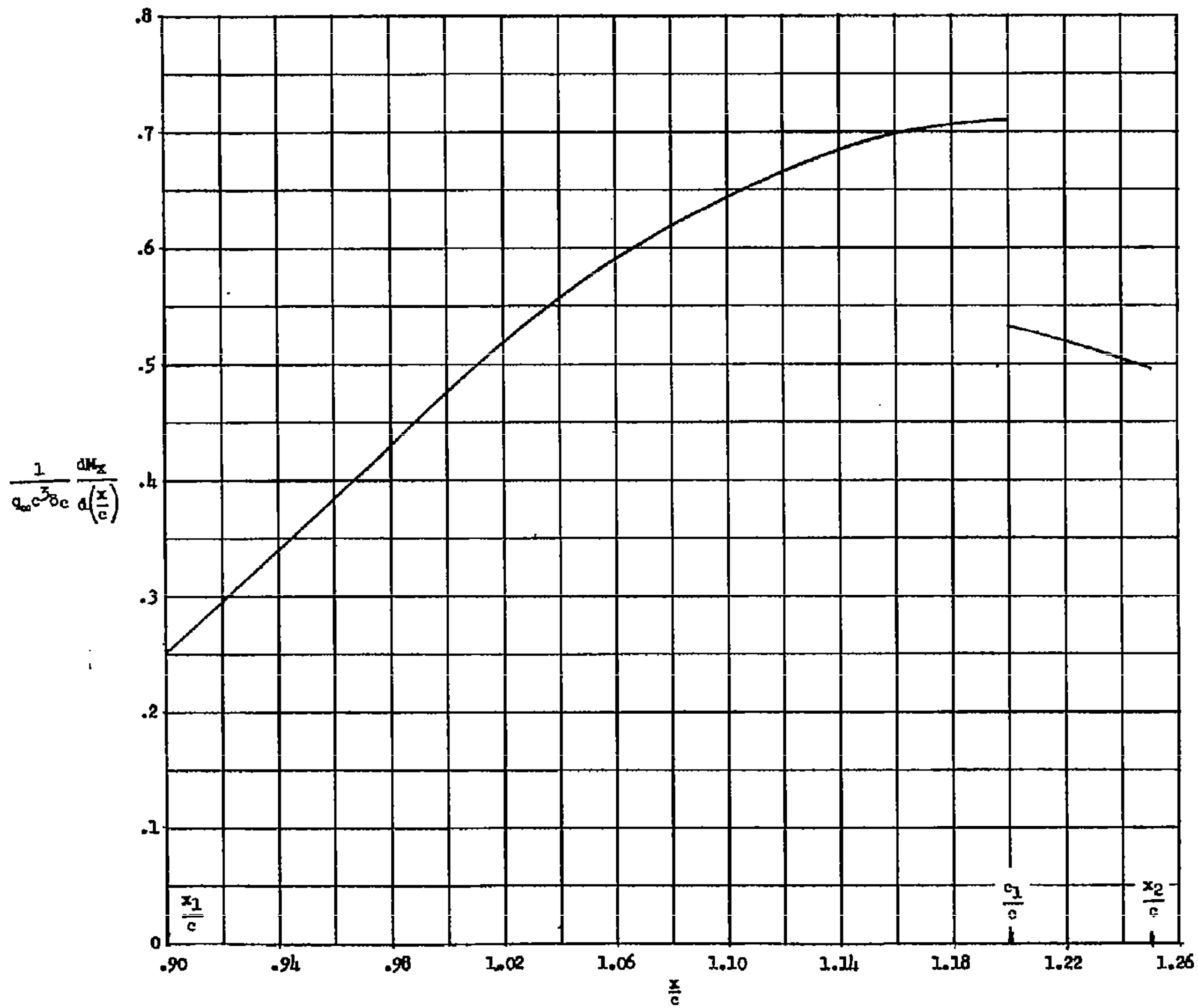


Figure 3.- Rolling-moment distribution due to aileron deflection.

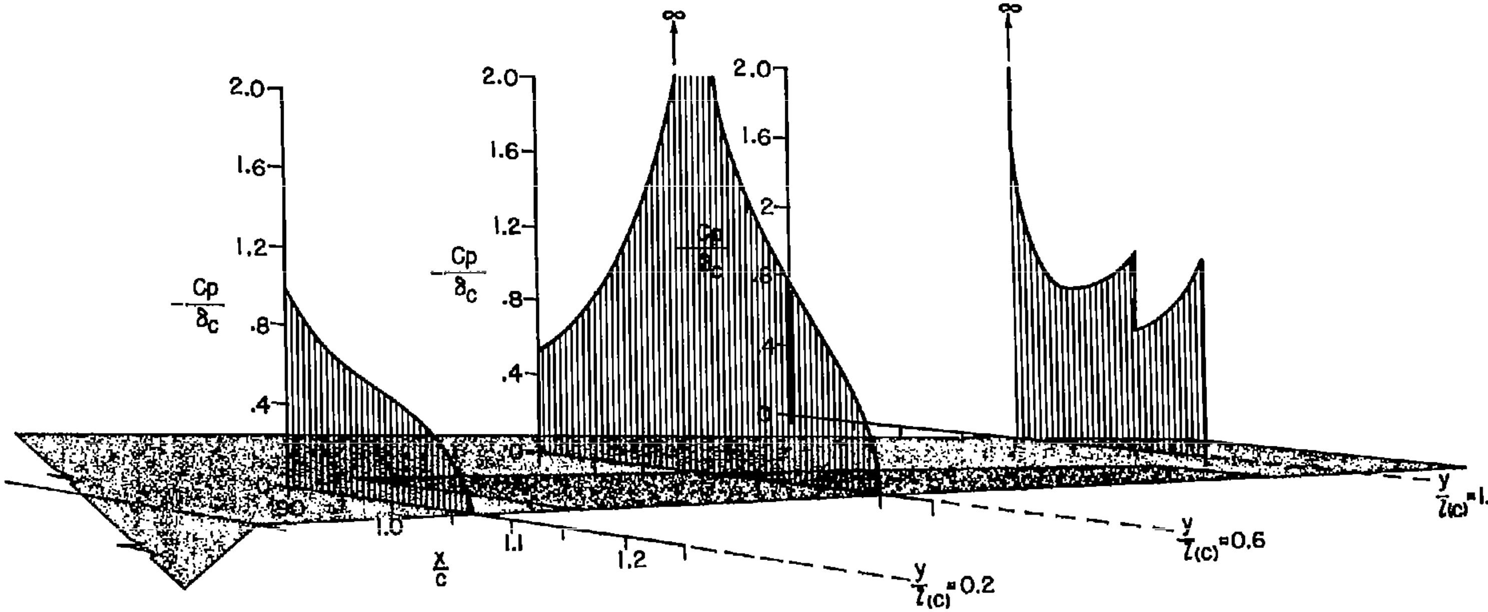


Figure 4.- Chordwise pressure distribution due to deflection of an aileron.